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OF THE RUSSIAN FEDERATION

FEDERAL STATE AUTONOMOUS EDUCATIONAL  
INSTITUTION OF HIGHER EDUCATION  
"NORTH CAUCASUS FEDERAL UNIVERSITY"

**SUGGESTIONS FOR TEACHING**  
**for the performance of practical work**  
**discipline "Automatic Control Theory"**  
for students of the field of study

11.03.02 Infocommunication Technologies and Communication Systems,  
Specialization "Intelligent Data Processing in Infocommunication Systems and  
Networks"

Stavropol, 2026  
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## **INTRODUCTION**

### **Purpose and objectives of mastering the discipline (module)**

The purpose of mastering the discipline is to study the basics and principles of control of automatic systems, the principles of building control systems with computers, the basics of digital signal processing.

### **Objectives of mastering the discipline:**

Study of the basic principles of control and system organization, mathematical foundations of models of continuous and discrete control systems, methods of analysis and synthesis of control systems, principles of building control systems with computers as control devices, basics of digital signal processing, as well as principles and methods of software implementation of control algorithms in digital systems.

### **The place of the discipline (module) in the structure of the educational program**

The discipline refers to the part formed by the participants of educational relations of the block "Block 1. Disciplines (modules)". Its development takes place in the 5th semester.

## PRACTICAL WORK

### Forms of representation of models of automatic control systems. Connection of the differential equation with the structural diagram of the system

Purpose of the lesson:

1. To study models for describing signals and continuous linear systems using ordinary differential equations.
2. Learn how to draw up a structural diagram of the system according to a given differential equation.
3. To learn how to make a differential equation of a system according to a given structural scheme.

Practical work issues:

1. Description of signals and continuous linear systems using ordinary differential equations.
2. Construction of a structural scheme based on a differential equation.
3. Compilation of a differential equation according to a structural scheme.

1. Description of signals and continuous linear systems using ordinary differential equations

Description of signals. Signals operating in control systems in the time domain are described by various functions, including generalized ones. Two typical signals are distinguished: impulse action, which is described by the delta function  $\delta(t - \tau)$ , and a single step function  $1(t - \tau)$ .

A unit pulse  $\delta(t)$  is a mathematical idealization of an extremely short pulse signal (Fig. 1a).

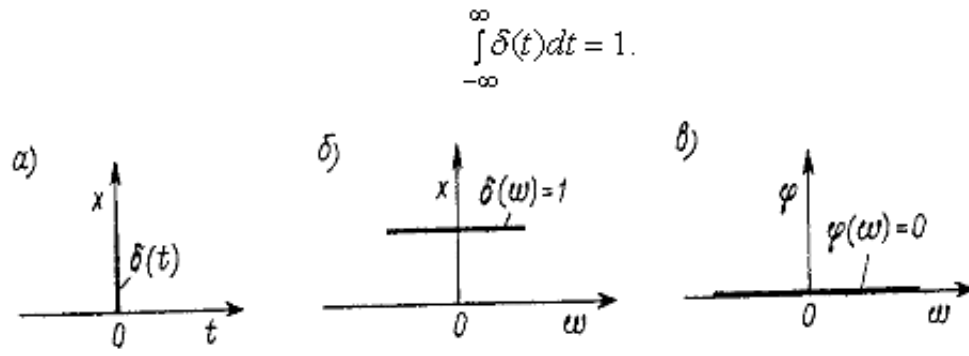


Figure 1a – Geometric interpretation: unit momentum a) and its amplitude b) and phase c) spectra

Interpretation of the effect of a single pulse: a short-term impact of the load on the motor shaft, short-circuit current, etc. At the same time, since the pulse duration is very short compared to the duration of the transient process  $t_{pp}$ , the real pulse can be represented with a high degree of accuracy as a single pulse with a certain scale factor.

A unit momentum  $\delta(t)$  is a momentum whose area is one with a duration equal to zero and a height equal to infinity

In Fig. 1a, the unit pulse is conventionally shown as a thickening on the ordinate axis and, accordingly, the amplitude and phase spectra of a single pulse signal are shown.

The step function and the delta function belong to the class of generalized functions, the first derivatives of which are delta functions.

The analytical expression for the impulse transient response, the impulse transient function of the link or the weight function of the link (the weight function), is denoted by  $w(t)$ . The expression for the unit pulse is respectively called the unit impulse function or the delta function  $\delta(t)$ .

$$\text{Thus, } w(t) \text{ is } y(t) \text{ at } x(t) = \delta(t) = \begin{cases} \infty, & t = 0; \\ 0, & t > 0, t < 0. \end{cases}$$

If  $x(t)$  is a non-unit momentum function, i.e.  $x(t) = k\delta(t)$ , then  $y(t) = kw(t)$ .

The delta function (asymmetric) is defined by the formula

$$\int_a^b f(t) \delta(t - \tau) dt = \begin{cases} f(\tau + 0), & \tau \in [a, b), \\ 0, & \tau \in (-\infty, a) \cup [b, +\infty), \end{cases} \quad (1)$$

valid for any piecewise continuous function  $f(t)$ . The derivatives of the delta function are defined in the same way:

$$\int_a^b f(t) \delta^{(k)}(t - \tau) dt = \begin{cases} (-1)^k f^{(k)}(\tau + 0), & \tau \in [a, b), \\ 0, & \tau \in (-\infty, a) \cup [b, +\infty), \end{cases}$$

where  $f(t)$  is any function that has a piecewise continuous derivative of the corresponding order.

### Single Step Function

$$1(t - \tau) = \begin{cases} 1, & t > \tau, \\ 0, & t \leq \tau. \end{cases} \quad (2)$$

The moment  $\tau$  corresponds to the moment of application of the input action to the control system (Fig. 1b).

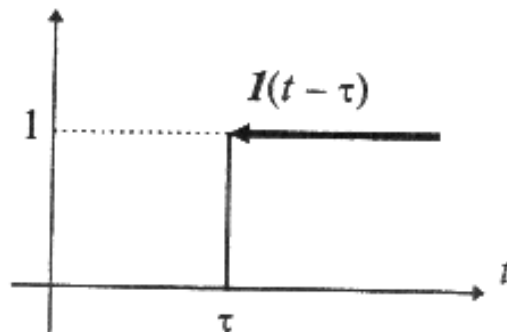


Figure 1b – Unit step function

Typical signals are related by a ratio

$$\int_{-\infty}^t \delta(\eta - \tau) d\eta = 1(t - \tau)$$

That is, the delta function  $\delta(t - \tau)$  can be considered a derivative of the unit step function  $1(t - \tau)$ .

Description of systems. Continuous processes occurring in control systems can be described by ordinary differential equations with corresponding initial conditions. Then, if the input signal is known, the output signal is determined as a result of solving the Cauchy problem for an ordinary differential equation.

A one-dimensional linear continuous nonstationary control system is described by a differential equation

$$a_n(t) \frac{d^n x(t)}{dt^n} + \dots + a_0(t)x(t) = b_m(t) \frac{d^m g(t)}{dt^m} + \dots + b_0(t)g(t) \quad (3)$$

with initial conditions

$$x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0, \dots, x^{(n-1)}(t_0) = x_0^{(n-1)}, \quad (4)$$

where  $g(t)$  is the input signal;  $x(t)$  is the output signal;  $t$  is time;  $a_n(t), \dots, a_0(t), b_m(t), \dots, b_0(t)$  are the coefficients of the left and right parts of the equation;  $n$  and  $m$  are the orders of the highest derivatives of the output and input signals, respectively;  $t_0$  is the moment of the beginning of the system's operation.

If the coefficients of the equation are constant, the system is called a linear stationary system:

$$a_n \frac{d^n x(t)}{dt^n} + \dots + a_0 x(t) = b_m \frac{d^m g(t)}{dt^m} + \dots + b_0 g(t) \quad (5)$$

In operator form, equation (3) has the form

$$D(p, t)x(t) = M(p, t)g(t)$$

where  $p = \frac{d}{dt}$  is the symbol denoting the differentiation operation;

$D(p, t), M(p, t)$  - differential operators of the left and right parts of the equation (3):

$$D(p, t) = a_n(t)p^n + \dots + a_1(t)p + a_0(t), \quad M(p, t) = b_m(t)p^m + \dots + b_1(t)p + b_0(t).$$

Equation (5) in operator form has the form

$$D(p)x(t) = M(p)g(t),$$

where

$$D(p) = a_n p^n + \dots + a_1 p + a_0, \quad M(p) = b_m p^m + \dots + b_1 p + b_0.$$

From the operator form of the equation follows the method of depicting a stationary system on structural diagrams (Fig. 2).

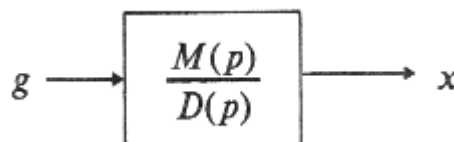


Figure 2. – Image of a stationary system on structural diagrams

## 2. Construction of a structural scheme based on a differential equation.

Structural diagrams are built using elementary, typical links and adders that describe the conversion of signals. They serve as one of the languages for describing control systems. Structural diagrams, as a rule, find an equivalent operator of the control system, and then solve various analysis problems.

### Algorithm for building a structural diagram

1. Express the term with the highest derivative of the differential equation and represent the resulting relation using the adder, differentiating and intensifying links.
2. All lower derivatives should be received as signals at the corresponding outputs of the integrating links connected in series.
3. The initial conditions should be presented as constant in time effects applied at the outputs of the integrating links.

**Example. Construct** a structural diagram of a system described by a differential equation

$$4\ddot{x} - 3\dot{x} + x = 2\dot{g}$$

with initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0.$$

Let us express the term with the highest derivative from the equation:

$$4\ddot{x} = 2\dot{g} + 3\dot{x} - x.$$

Let us draw a diagram of signal acquisition  $4\ddot{x}$ . With the help of an amplifying link with a gain we 1/4 obtain a signal  $\ddot{x}$ . Let us construct a straight circuit of the circuit by converting the signal sequentially  $\ddot{x}$  by the integrating links. By adding the appropriate initial conditions to the outputs of the integrating links, we obtain a part of the direct circuit of the circuit in which the output signal  $x$  and its derivatives  $\dot{x}$ ,  $\ddot{x}$  are present .  $4\ddot{x} = 2\dot{g} + 3\dot{x} - x$ .

To do this, we add to the straight circuit the connection of the differentiating and amplifying links, which from the input signal  $g$  allow you to get the desired signal  $2\dot{g}$  at the input of the combiner. The signals  $x$  are  $3\dot{x}$  fed to the combiner with the appropriate sign, using feedback.

Thus, we obtain a structural scheme corresponding to a given differential equation (Fig. 3).

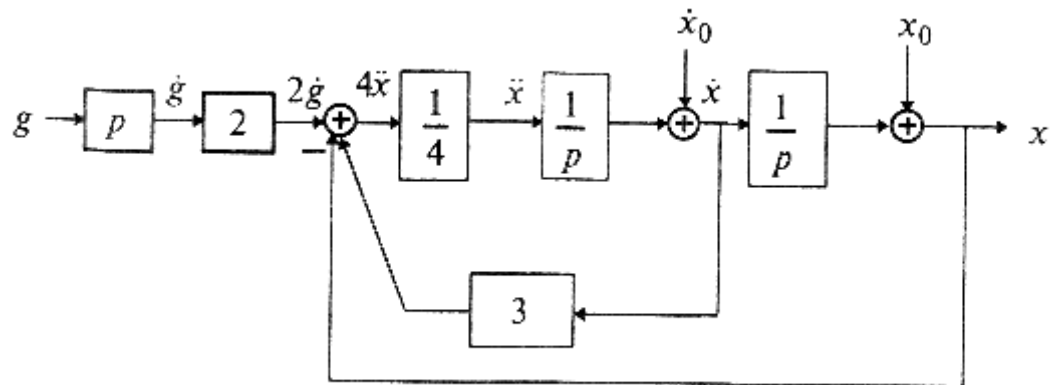


Figure 3. – Structural diagram of the system

### 3. Compilation of a differential equation according to a structural scheme

#### Algorithm for compiling a differential equation according to a structural scheme

To write the differential equation, you should:

- mark all intermediate signals on the diagram;
- write down the equations for each link and for each adder;
- exclude intermediate variables from the resulting system of differential and algebraic equations, except for the input and output signals.

**Example.** Make a differential equation according to the structural scheme shown in Fig. 4.

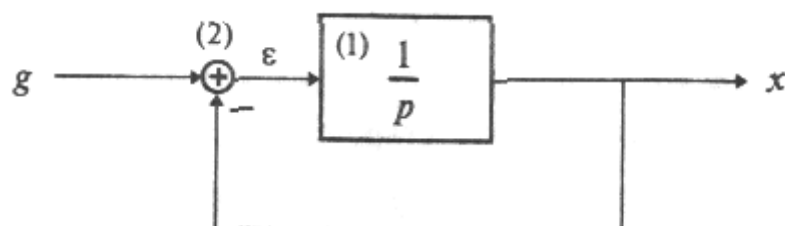


Figure 4. - Structural diagram of the system

Let's make the equations of the circuit elements:

$$1) \quad x = \frac{1}{p} \varepsilon; \quad 2) \quad \varepsilon = g - x.$$

$$\text{Hence } x = \frac{1}{p}(g - x), \quad px = g - x, \quad (p+1)x = g.$$

The differential equation of the system has the form

$$\dot{x}(t) + x(t) = g(t)$$

**Example.** Make a differential equation according to the structural scheme of Fig.5.

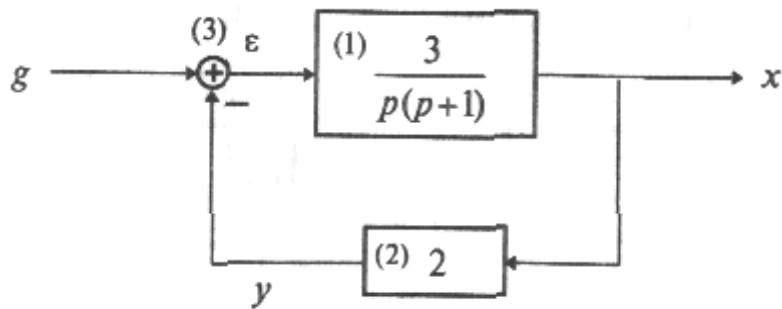


Figure 5. – Structural diagram of the system

Let's make the equations of the circuit elements:

$$1) \quad x = \frac{3}{p(p+1)} \varepsilon; \quad 2) \quad y = 2x; \quad 3) \quad \varepsilon = g - y.$$

Hence

$$(p^2 + p)x = 3(g - y) = 3(g - 2x) = 3g - 6x.$$

Moving from the operator form of notation of a differential equation to the ordinary one, we get

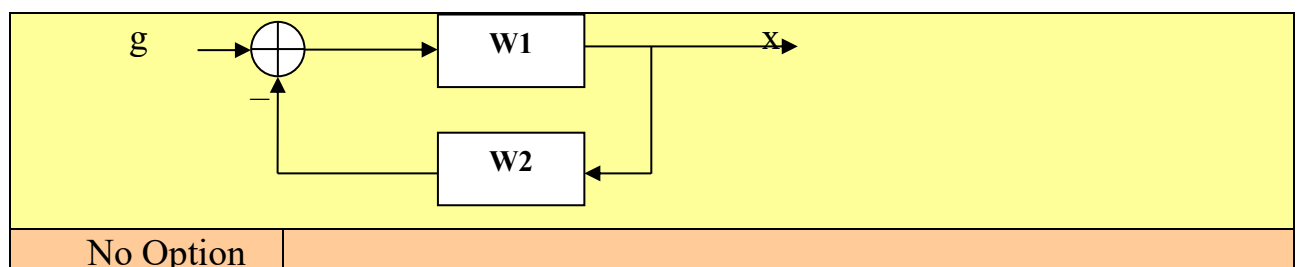
$$\ddot{x} + \dot{x} + 6x = 3g.$$

### Practical part of the lesson

1. To construct structural diagrams of systems given by the following differential equations under the following initial conditions  $x'(0) = x'_0$ ,  $x(0) = x_0$ .

| No Option | Equation                         |
|-----------|----------------------------------|
| 1.        | $2x''-3x'+4x=3g'+g$              |
| 2.        | $4x'''-6x''+x=2g'+3g$            |
| 3.        | $x''+2x=5g$                      |
| 4.        | $3x''+4x'+6x=2g$                 |
| 5.        | $2x''+6x'+x=3g'+2g$              |
| 6.        | $x''' + x' + x = g$              |
| 7.        | $5x''' + tx'' + t^2x = g' + 2g$  |
| 8.        | $4x''-3x'+x=2g$                  |
| 9.        | $x''-3x'+x=g'+g$                 |
| 10.       | $x''-3x'-4x=3g'-3g$              |
| 11.       | $x''-x'+4x=g'+g$                 |
| 12.       | $7x''' + 6x'' + 8x = g' + g$     |
| 13.       | $5x'' - 12x = 11g$               |
| 14.       | $3x'' + x' + x = g$              |
| 15.       | $2x'' + x' + 3x = 3g' + g$       |
| 16.       | $x''' + x' + x = g$              |
| 17.       | $5x''' - tx'' + t^2x = 4g' + 2g$ |
| 18.       | $3x'' + 2x' + x = 2g$            |
| 19.       | $x'' - 3x' + x = g' + g$         |
| 20.       | $x'' - 3x' - 4x = 3g' - 3g$      |
| 21.       | $2x'' - 3x' + 4x = 3g' + g$      |
| 22.       | $4x''' - 6x'' + x = 2g' + 3g$    |
| 23.       | $6x'' + x' + 2x = 5g$            |
| 24.       | $3x'' + 4x' + 6x = 2g$           |
| 25.       | $2x'' + 6x' + x = 3g' + 2g$      |
| 26.       | $x''' + x' + x = g$              |
| 27.       | $x''' + tx'' + t^2x = g' + g$    |
| 28.       | $4x'' + x' + x = g$              |
| 29.       | $x'' - 3x' + x = g' + g$         |
| 30.       | $x'' - 3x' - 4x = g' - g$        |

2. To determine the differential equations of the systems according to the given structural schemes



|     |                                                      |
|-----|------------------------------------------------------|
| 1.  | $W_1 = \frac{1}{p+2k}, \quad W_2 = 1$                |
| 2.  | $W_1 = \frac{1}{Tp+1}, \quad W_2 = 3k$               |
| 3.  | $W_1 = \frac{1}{Tp-1}, \quad W_2 = 2$                |
| 4.  | $W_1 = \frac{1}{Tp+1}, \quad W_2 = \frac{1}{p}$      |
| 5.  | $W_1 = \frac{1}{Tp-1}, \quad W_2 = \frac{1}{p}$      |
| 6.  | $W_1 = \frac{1}{Tp+1}, \quad W_2 = k$                |
| 7.  | $W_1 = \frac{1}{Tp-1}, \quad W_2 = k$                |
| 8.  | $W_1 = \frac{1}{p}, \quad W_2 = k$                   |
| 9.  | $W_1 = \frac{k}{p+1}, \quad W_2 = 2k$                |
| 10. | $W_1 = \frac{2}{Tp+1}, \quad W_2 = 4k$               |
| 11. | $W_1 = \frac{k}{(p^2+p+1)}, \quad W_2 = kp$          |
| 12. | $W_1 = \frac{k}{p(Tp+1)}, \quad W_2 = 1$             |
| 13. | $W_1 = \frac{k}{p(Tp-1)}, \quad W_2 = 3$             |
| 14. | $W_1 = \frac{k}{(p^2+p+1)}, \quad W_2 = \frac{1}{p}$ |
| 15. | $W_1 = \frac{k}{T^2p^2+1}, \quad W_2 = 1$            |
| 16. | $W_1 = \frac{k}{T^2p^2-1}, \quad W_2 = k$            |
| 17. | $W_1 = \frac{k}{T^2p^2-1}, \quad W_2 = \frac{1}{p}$  |
| 18. | $W_1 = \frac{k}{p^2-1}, \quad W_2 = \frac{k}{p}$     |
| 19. | $W_1 = \frac{k}{(T_2^2p^2+T_1p+1)}, \quad W_2 = 1$   |
| 20. | $W_1 = \frac{k}{(p^2-p+1)}, \quad W_2 = kp$          |
| 21. | $W_1 = \frac{k}{(T_2^2p^2-T_1p-1)}, \quad W_2 = 1$   |

|     |                                                          |
|-----|----------------------------------------------------------|
| 22. | $W_1 = \frac{k}{(p^2 + 1)}, \quad W_2 = \frac{k}{p}$     |
| 23. | $W_1 = \frac{k}{(T_2^2 p^2 - T_1 p + 1)}, \quad W_2 = 1$ |
| 24. | $W_1 = \frac{k}{(T_2^2 p^2 - T_1 p - 1)}, \quad W_2 = 1$ |
| 25. | $W_1 = kp, \quad W_2 = k$                                |
| 25. | $W_1 = kp, \quad W_2 = \frac{1}{p}$                      |
| 27. | $W_1 = kp, \quad W_2 = \frac{k}{p(Tp + 1)}$              |
| 28. | $W_1 = \frac{1}{p}, \quad W_2 = kp$                      |
| 29. | $W_1 = \frac{k}{p(Tp - 1)}, \quad W_2 = kp$              |
| 30. | $W_1 = \frac{1}{p}, \quad W_2 = \frac{k}{p(Tp - 1)}$     |

The report of practical work should reflect the answers to the control questions.

Control questions:

1. How is the delta function described? Describe it.
2. Characterize a single step function.
3. How is a one-dimensional linear continuous nonstationary system described by a differential equation?
4. What is the difference in the description of a continuous non-stationary system and a linear stationary system?
5. How is a stationary system depicted on a structural diagram?
6. Give an algorithm for constructing a structural diagram based on the differential equation of the system.
7. How to obtain a description of a system by a differential equation according to a given structural scheme?



## **PRACTICAL WORK**

### **Study of the timing characteristics of self-propelled guns**

#### **Purpose of the work:**

1. To study the dynamic characteristics of automatic control systems (ACS) using the Appendix to Practical Work
2. Acquire skills in obtaining and studying linear dynamic models using the Control System Toolbox of the Matlab system.

#### **Study questions:**

1. Study of the transient functions of ACS,
2. Study of impulse transient functions of ACS.

#### **Problem statement**

The object of the study is linear (linearized) dynamic stationary control systems with one input and one output. At the same time, the model of a one-dimensional ACS is defined as a complex transfer function recorded as a ratio of polynomials

$$W(s) = \frac{b_m s^m + \dots + b_1 s + b_0}{a_n s^n + \dots + a_1 s + a_0}.$$

It is necessary to obtain and study the time characteristics of the SPG for your variant.

#### **Brief information from the theory**

Description of systems. Continuous processes occurring in control systems can be described by ordinary differential equations with corresponding initial conditions. Then, if the input signal is known, the output signal is determined as a result of solving the Cauchy problem for an ordinary differential equation.

A one-dimensional linear continuous nonstationary control system is described by a differential equation

$$a_n(t) \frac{d^n x(t)}{dt^n} + \dots + a_0(t) x(t) = b_m(t) \frac{d^m g(t)}{dt^m} + \dots + b_0(t) g(t) \quad (1)$$

with initial conditions

$$x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0, \dots, x^{(n-1)}(t_0) = x_0^{(n-1)}, \quad (2)$$

where  $g(t)$  is the input signal;  $x(t)$  is the output signal;  $t$  is the time;  $a_n(t), \dots, a_0(t), b_m(t), \dots, b_0(t)$  is the coefficients of the left and right parts of the equation;  $n$  and  $m$  is the order of the highest derivatives of the output and input signals, respectively;  $t_0$  is the moment when the system starts functioning.

If the coefficients of the equation are constant, the system is called a linear stationary system:

$$a_n \frac{d^n x(t)}{dt^n} + \dots + a_0 x(t) = b_m \frac{d^m g(t)}{dt^m} + \dots + b_0 g(t) \quad (3)$$

In operator form, equation (1) has the form

$$D(p, t)x(t) = M(p, t)g(t)$$

where  $p = \frac{d}{dt}$  is the symbol denoting the differentiation operation;  $D(p, t), M(p, t)$  are the differential operators of the left and right sides of the equation (3):

$$D(p, t) = a_n(t)p^n + \dots + a_1(t)p + a_0(t), \quad M(p, t) = b_m(t)p^m + \dots + b_1(t)p + b_0(t).$$

Equation (3) in operator form has the form

$$D(p)x(t) = M(p)g(t),$$

where

$$D(p) = a_n p^n + \dots + a_1 p + a_0, \quad M(p) = b_m p^m + \dots + b_1 p + b_0.$$

From the operator form of the equation follows the method of depicting a stationary system on structural diagrams (Fig. 1).

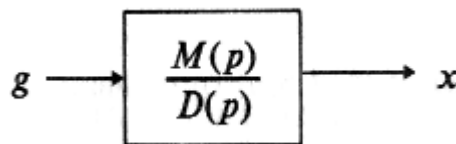


Figure 1. – Image of a stationary system on structural diagrams

Let us consider an automatic control system (ACS) described by a linear (linearized) differential equation of the form:

$$\begin{aligned} a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = \\ = b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \dots + b_1 \frac{du(t)}{dt} + b_0 u(t), \end{aligned} \quad (4)$$

where  $u(t)$  is the input process,  $y(t)$  is the output process,  $a_i, b_j, (i = \overline{0, n}, j = \overline{0, m})$  are constant coefficients,  $n, m$  ( $n \geq m$ ) are constant numbers. In the operator form, expression (4) can be written –

$$A(D)y(t) = B(D)u(t).$$

Here  $D$  is the differentiation operator  $\left(D \stackrel{\text{def}}{=} \frac{d}{dt}\right)$ . Hence the "input-output" transformation of the system –

$$\frac{y(t)}{u(t)} = \frac{B(D)}{A(D)} = W(D), \quad (5)$$

where  $W(D)$  is called the operator transfer function.

One way to model systems is to represent the input-output transformation as a complex transfer function:

$$\frac{y(s)}{u(s)} = \frac{B(s)}{A(s)} = W(s), \quad (6)$$

which is obtained by applying the Laplace transform to (5) under initial zero conditions. Here  $s$  is the complex variable. The relationship between the operator (5) and complex (6) transfer functions can be written as

$$W(s) = W(D)|_{D=s}.$$

**The transfer function is equal to the Laplace ratio of the input and output variables under zero initial conditions**, where the integral Laplace transform is defined as:

$$y(s) = L\{y(t)\} = \int_0^{\infty} y(t)e^{-st} dt;$$

$$u(s) = L\{u(t)\} = \int_0^{\infty} u(t)e^{-st} dt.$$

Thus, knowing the differential equation of the system, it is possible to write down its transfer function, and, conversely, it is possible to write the differential equation from the transfer function.

Complex numbers that are the roots of the polynomial  $B(s)$  are called **zeros** of the transfer function, and the roots of the polynomial  $A(s)$  are **called poles**.

The explicit type of input and output relationship is defined by the expression:

$$y(t) = \int_0^t w(t)v(t-\tau)d\tau, \quad (7)$$

where  $w(t)$  is the original (i.e., obtained by the inverse Laplace transformation) of the complex transfer function  $W(s)$ .

Dynamic properties of systems characterize reactions to input impacts of a special type.

Time characteristics are one of the forms of representation by operators of converting an input signal variable into an output signal variable. In particular, the analysis of the system's output to a unit hop and  $\delta$ an -function (delta function).

Let  $u(t) = 1(t)$ , that is, a signal is fed to the input of the system, described by the Heaviside function (single hop), defined by

$$1(t) = \begin{cases} 0, & \text{при } t \leq 0, \\ 1, & \text{при } t > 0. \end{cases}$$

A graph of the Heaviside function is shown in Fig. 2.

**The response of the ACS to a single jump is called the transient function of the system at zero initial conditions and is denoted  $h(t)$ .**

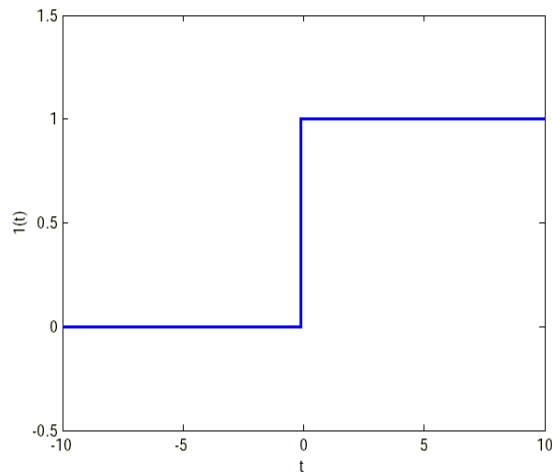


Figure 2. – Heaviside function (unit jump)

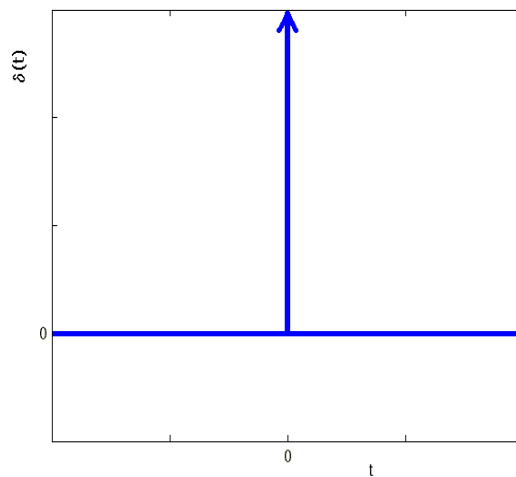


Figure 3. – Dirac function (momentum function)

If  $u(t) = \delta(t)$ , that is, the input of the system receives a signal determined by the Dirac function ( $\delta$ -function, impulse function), Fig. 3

$$\delta(t) = \begin{cases} \infty, & \text{при } t = 0, \\ 0, & \text{при } t \neq 0, \end{cases}$$

then the **reaction of the ACS to the impulse action is called the pulse transient function of the system and is designated  $w(t)$ .**

In this way, the original complex transfer function can be measured as the response of the system to the momentum.

The pulse transient and transient functions of the system are related by the relation (of (7)):

$$h(t) = \int_0^t w(\tau) d\tau.$$

### Sequence of work

To perform practical work, the Control System Toolbox application package (PPP) is used, which is designed to work with LTI (Linear Time Invariant Models) models of control systems.

Control System Toolbox has a data type that defines a dynamic system as a complex transfer function. The command syntax that creates an LTI system with one input and one output as a transfer function:

$$\text{TF}([bm, \dots, b1, b0], [an, \dots, a1, a0]),$$

$bm, \dots, b1$  are the values of the coefficients of polynomial  $B$  in (3),  $an, \dots, a1$  are the values of the coefficients of polynomial  $A$  in (3).

The commands given in Table 1 can be used to perform the work.

Table 1. – Some Control System Toolbox Commands

| Syntax                                    | Description                                                                                                     |
|-------------------------------------------|-----------------------------------------------------------------------------------------------------------------|
| <code>pole(&lt;LTI-object&gt;)</code>     | Calculating the Poles of the Transfer Function                                                                  |
| <code>zero(&lt;LTI-объект&gt;)</code>     | Calculation of Zeros of the Transfer Function                                                                   |
| <code>K=dcgain(&lt;LTI-объект&gt;)</code> | Calculates the transfer factor k                                                                                |
| <code>step(&lt;LTI-объект&gt;)</code>     | Construction of a transient process graph. Calculates and plots the model's response to a single step function. |
| <code>impulse(&lt;LTI-объект&gt;)</code>  | Plotting the Pulse Transient Function                                                                           |

To determine the roots of degree  $k$  polynomials, the MATLAB **roots(P)** **command can be used**, which, as an argument  $P$ , receives a matrix of polynomial coefficients  $[pk, \dots, p0]$ .

Another option for obtaining graphs of the dynamic characteristics of ACS is to use the CT – LTI viewer graphical interface of the PPP, which is called by the **ltiview** command, which has the name of the variable containing the LTI object as a parameter.

As an example, let's look at the use of the basic commands of the Control System Toolbox.

### Methodical example

The transfer function of the ACS is specified

$$W = \frac{s + 2}{3s^3 + 4s^2 + 5s + 3}.$$

It is necessary to determine the timing characteristics of the system.

We will work in the command mode of the MATLAB environment.

1. Create an LTI object named w, to do this:

```
>> w=tf([1 2],[3 4 5 3])
```

Transfer function:

```
      s + 2
-----
3 s^3 + 4 s^2 + 5 s + 3
```

2. Define the poles and zeros of the transfer function using the pole, zero commands.

```
>> pole(w)
```

ans =

```
-0.2639 + 1.0825i
-0.2639 - 1.0825i
-0.8055
```

```
>> zero(w)
```

ans =

```
-2
```

3. On the complex plane, determine the location of the poles and zeros for a given transfer function using the pzmap (w) command in Fig. 4.

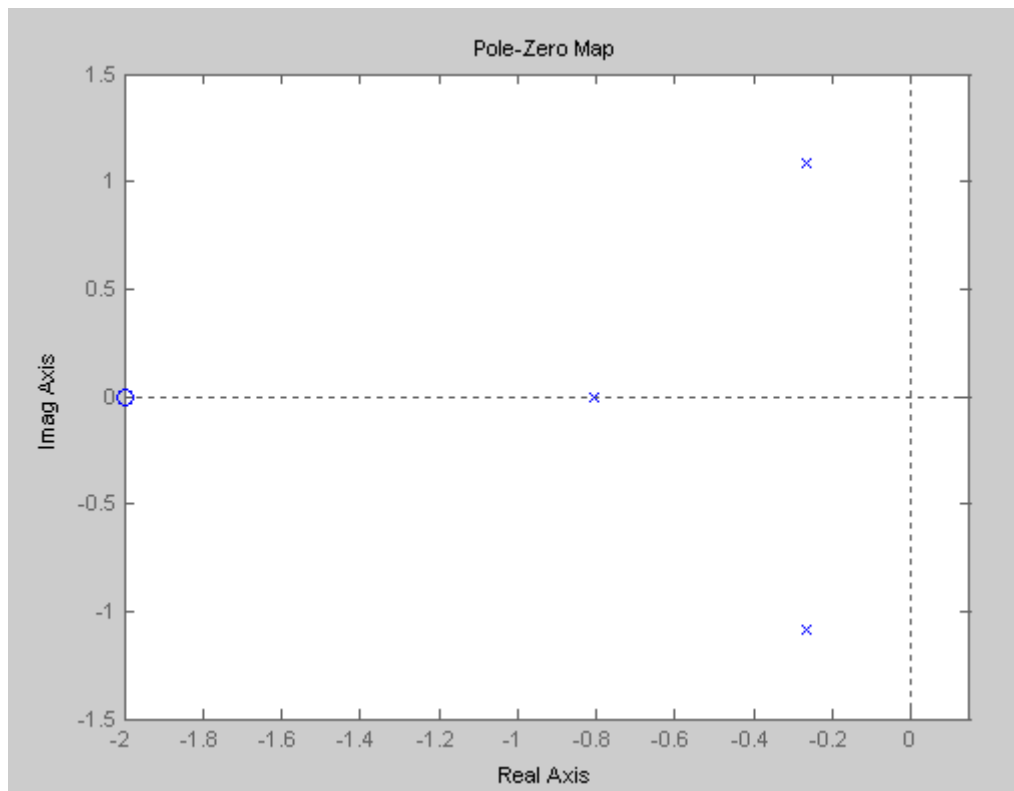


Figure 4. – Location of Poles and Units for a given Transfer Function

4. Define the transient function using the `step(w)` command. The result of its execution is shown in Fig. 5.

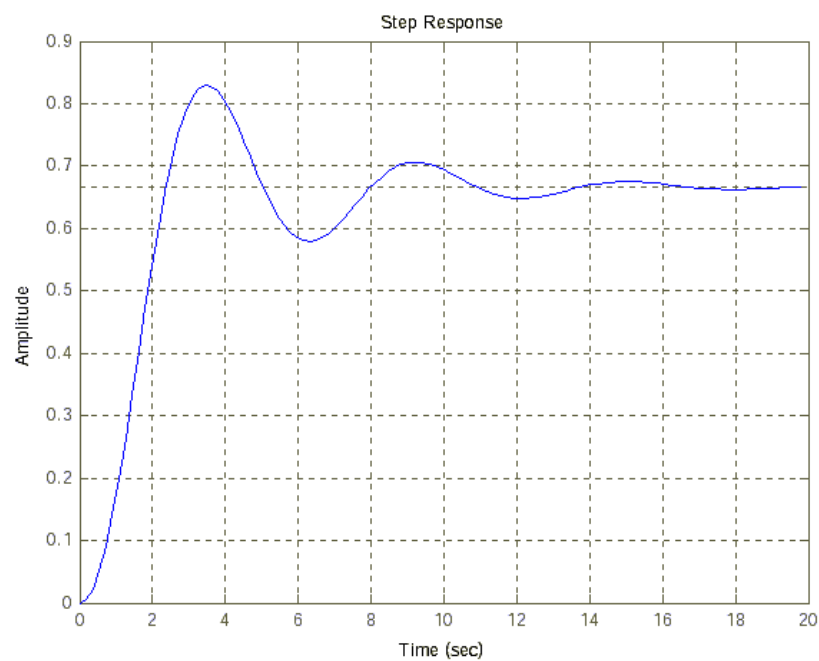


Figure 5. – Transient function  $h(t)$

5. Construct a pulse transient function using the `impulse(w)` command. The result is shown in Fig. 6.

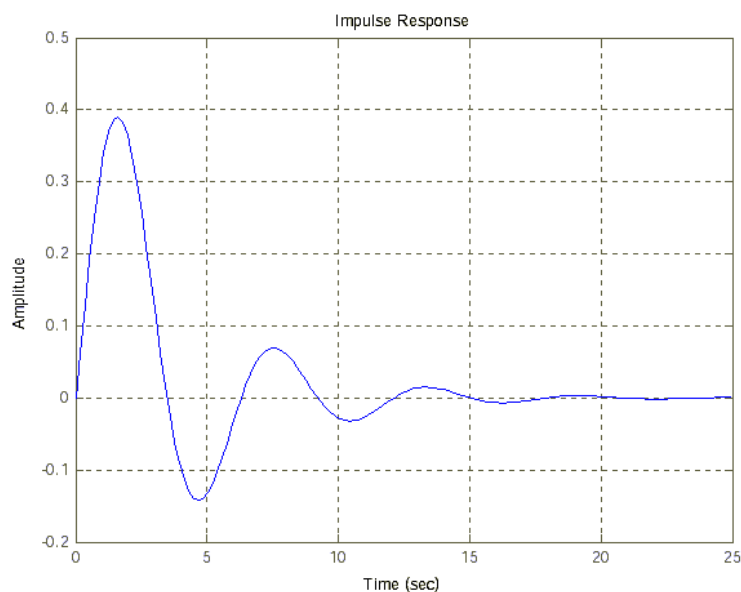


Figure 6. – Pulse transient function

6. Calculate the transfer factor using the command `k=dcgain(w)`.

`>> k=dcgain(w)`

`k =`

0.6667

### Variants of tasks

| №  | Type of transfer function                                              | №  | Coefficients of polynomials |       |       |       |       |       |       |
|----|------------------------------------------------------------------------|----|-----------------------------|-------|-------|-------|-------|-------|-------|
|    |                                                                        |    | $b_0$                       | $b_1$ | $a_0$ | $a_1$ | $a_2$ | $a_3$ | $A_4$ |
| 1. | $W(p) = \frac{b_1 p + b_0}{a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0}$ | 1. | 0                           | 3     | 1     | 2     | 3     | 0     | 1     |
|    |                                                                        | 2. | 2                           | 6     | 4     | 0     | 1     | 5     | 1     |
|    |                                                                        | 3. | 0                           | -3    | 5     | 2     | 0     | 2     | 1     |
|    |                                                                        | 4. | 4                           | 2     | 3     | 4     | 5     | 3     | 1     |
|    |                                                                        | 5. | 0                           | 1     | -2    | -2    | -3    | -2    | 0     |
|    |                                                                        |    | $b_0$                       | $b_1$ | $b_2$ | $a_0$ | $a_1$ | $a_2$ | $A_3$ |
| 2. | $W(p) = \frac{b_2 p^2 + b_1 p + b_0}{a_3 p^3 + a_2 p^2 + a_1 p + a_0}$ | 1. | 0                           | -3    | 2     | 4     | 2     | 3     | 9     |
|    |                                                                        | 2. | 8                           | 0     | -3    | -4    | -6    | -4    | -1    |
|    |                                                                        | 3. | -4                          | 6     | -2    | 5     | 5     | 0     | 1     |
|    |                                                                        | 4. | 6                           | -8    | -7    | 0     | -6    | -3    | -1    |
|    |                                                                        | 5. | 2                           | -1    | -3    | -1    | 0     | -7    | -2    |
|    |                                                                        |    | $b_0$                       | $b_1$ | $b_2$ | $a_0$ | $a_1$ | $a_3$ | $a_4$ |

| №  | Type of transfer function                                                    | №  | Coefficients of polynomials |       |       |       |       |       |       |
|----|------------------------------------------------------------------------------|----|-----------------------------|-------|-------|-------|-------|-------|-------|
|    |                                                                              |    | $b_0$                       | $b_1$ | $a_0$ | $a_1$ | $a_2$ | $a_3$ | $A_4$ |
| 3. | $W(p) = \frac{b_2 p^2 + b_1 p + b_0}{a_4 p^4 + a_3 p^3 + a_1 p + a_0}$       | 1. | 0                           | 2     | 8     | -3    | 7     | -7    | 1     |
|    |                                                                              | 2. | -5                          | 0     | 3     | -8    | -2    | -1    | -6    |
|    |                                                                              | 3. | -7                          | 1     | 2     | 0     | 5     | 2     | 9     |
|    |                                                                              | 4. | -6                          | 4     | -4    | 1     | 0     | 6     | 3     |
|    |                                                                              | 5. | 2                           | -2    | -1    | 5     | 3     | 0     | 9     |
| 4. | $W(p) = \frac{b_2 p^2 + b_0}{a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0}$     | 1. | 0                           | -5    | 4     | 3     | 7     | 9     | 1     |
|    |                                                                              | 2. | 7                           | -6    | 0     | 5     | 8     | 2     | 2     |
|    |                                                                              | 3. | -2                          | -8    | 2     | 0     | 4     | 3     | 3     |
|    |                                                                              | 4. | -7                          | -1    | 6     | 9     | 0     | 4     | 2     |
|    |                                                                              | 5. | -3                          | 7     | -4    | 4     | 5     | 0     | 1     |
|    |                                                                              |    | $b_2$                       | $b_3$ | $a_0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
| 5. | $W(p) = \frac{b_3 p^3 + b_2 p^2}{a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0}$ | 1. | 0                           | -5    | 4     | 3     | 7     | 9     | 1     |
|    |                                                                              | 2. | 7                           | -6    | 0     | 5     | 8     | 2     | 2     |
|    |                                                                              | 3. | -2                          | -8    | 2     | 0     | 4     | 3     | 3     |
|    |                                                                              | 4. | -7                          | -1    | 6     | 9     | 0     | 4     | 2     |
|    |                                                                              | 5. | -3                          | 7     | -4    | 4     | 5     | 0     | 1     |

### Procedure for performing practical work

1. To study theoretical information.
2. Using a complex transfer function, compile a differential equation that determines the functioning of the ACS.
3. Use the differential equation to build a structural diagram of the ACS.
4. Start the MATLAB system.
5. Create a tf object according to the specified option.
6. Determine the poles of the transfer function  $s_j^*, (i = \overline{1, n})$  using the **pole** command.

7. Determine the zeros of the transfer function  $s_j^0, (j = \overline{1, m})$  using the command **zero**.

8. On the complex plane, obtain the location of the poles and zeros of the transfer function.

9. Calculate the transfer factor using the command  $k=dcgain(w)$ .

1. Using the LTI-viewer, or the corresponding commands (Table 1), obtain the time characteristics – the transient function  $h(t)$ , the pulse transient function  $w(t)$ . Describe the nature of the obtained time characteristics. Determine the overshoot, transient time

2. To analyze the results obtained for each studied characteristic and to draw quantitative and qualitative conclusions.

3. Answer control questions.

4. Draw up a report.

5. Submit the report to the teacher and defend the work.

The report is drawn up in accordance with the requirements for the design of laboratory work at the university, and must contain a title page, a statement of the purpose of the work, a statement of the task in accordance with the version of the task, the results of the work, conclusions, answers to control questions. Answers to control questions must be presented in the report.

Note: Variants of tasks consist of two digits: the first is the number of the transfer function, the second is the number of the set of coefficient values.

### **Control questions**

1. How are the continuous processes in control systems described?
2. Represent a one-dimensional linear, continuous, non-stationary, and stationary system using ordinary differential equations.
3. How are stationary systems depicted in structural diagrams?
4. Define the complex transfer function of the system.
5. How are zeros and poles of the transfer function determined? Explain their physical essence.

6. Define and explain the physical meaning of the transient function.
7. Define and explain the physical meaning of the momentum transient function.
8. How are the transient and impulse transient functions of the system related?
9. How is a single function with a delay described? Explain its physical meaning.
10. What is the relationship between the transition and weight functions of linear links?
11. Explain what is meant by the terms "original" and "image"

## **PRACTICAL WORK**

### **Frequency Response Study Self-propelled guns**

#### **Purpose of the work:**

1. To study the frequency characteristics of automatic control systems.
2. Acquire skills in obtaining and studying linear dynamic models using the Control System Toolbox of the Matlab system.

#### **Study questions:**

1. Amplitude and frequency characteristics of the ACS.
2. Phase frequency response of the ACS.
3. Logarithmic amplitude-frequency (LAFC) and phase-frequency characteristics (LFFS) of ACS.

#### **Problem statement**

The object of the study is linear (linearized) dynamic stationary control systems with one input and one output. At the same time, the model of a one-dimensional ACS is defined as a complex transfer function recorded as a ratio of polynomials

$$W(s) = \frac{b_m s^m + \dots + b_1 s + b_0}{a_n s^n + \dots + a_1 s + a_0}.$$

It is necessary to obtain and study the frequency characteristics of the ACS for your version.

#### **Brief information from the theory**

Due to the wide use in the study of the stability of dynamic systems and the design of regulators, frequency characteristics have become widespread.

The frequency characteristics of elements and systems are the dependence of the steady-state response parameters to harmonic signals of all frequencies and unit amplitudes. In linear systems, the form and frequency of the steady-state response coincide with the input waveform and frequency.

Let the input of the system with the transfer function  $W(s)$  a harmonic signal is given

$$u(t) = u_0 \cos(\omega t), t > 0. \quad (5)$$

Under these conditions, the following theorem is valid:

If the link is stable, then the steady-state reaction  $y(t)$  on the harmonic effect is a function of the same frequency with the amplitude

$$ay = au |W(i\omega)|,$$

and relative phase shift

$$\psi = \arg W(i\omega).$$

Thus, the output is determined by the harmonic function

$$y(t) = au |W(i\omega)| \cos(\omega t + \arg W(i\omega)),$$

where  $i$  is a complex unit ( $i^2 = -1$ ),  $W(i\omega) = W(s)|_{s=i\omega}$  is the frequency response.

Frequency Response  $W(i\omega)$  of a stationary dynamical system is called the Fourier transform of the transient function:

$$W(i\omega) = F[h(t, \tau)] = \int_0^{\infty} w(t - \tau) e^{-i\omega(t - \tau)} d\tau,$$

where  $w(t - \tau)$  is the momentum transient function.

The relationship between the complex transfer function and the frequency response, based on the properties of the Fourier transform, can be represented as a relation:

$$W(s)|_{s=i\omega} = W(i\omega).$$

With a fixed value  $\omega$  The frequency response is a complex number, and therefore can be represented as

$$W(i\omega) = A(\omega) e^{i\omega + \psi(\omega)} = U(\omega) + iV(\omega).$$

Here

$A(\omega) = |W(i\omega)|$  – amplitude-frequency response (AFC);

$\psi(\omega) = \arg W(i\omega)$  – phase-frequency response (PFC);

$U(\omega) = \operatorname{Re} W(i\omega)$  – real frequency response (HFC);

$V(\omega) = \text{Im } W(i\omega)$  – imaginary frequency response (MFR).

The complex frequency response  $W(i\omega)$  makes it possible to determine the amplitude  $A(\omega)$  and phase  $\psi(\omega)$  of the harmonic signal at the output of the system by the value of the frequency  $\omega$ . Real objects with an increase in frequency transmit signals worse – they weaken the amplitude and introduce a negative phase shift. Thus, the frequency response is located as a module, and the frequency response is found as an argument for the complex function of the real argument  $W(i\omega)$ .

Geometric place of points  $W(i\omega)$  on the complex plane when changing  $\omega$  from  $\omega_0$  up to  $\omega_1$  (usually  $\omega_0=0$ ,  $\omega_1=\infty$ ) is called the amplitude-phase response (APR) or Nyquist frequency hodograph.

The Nyquist frequency hodograph is used in the analysis of such properties of regulation and control systems as phase and amplitude stability, transmission coefficient, bandwidth, and system response to disturbances.

Using the Nichols hodograph and the Nichols diagram superimposed on it, the properties of both the open and closed models are analyzed. The `ngrid` command plots the constant amplitude and phase lines corresponding to the following transformation of the complex plane  $H$ :

$$W = \frac{H}{1+H}.$$

Such a coordinate grid is called a Nichols diagram. For one-dimensional models, it relates the Nichols hodograph points for an open system to the hodograph points for a closed unit negative feedback system.

The `ngrid` command builds a grid of coordinates for the module in the range from -40 to 40 dB, for the phase - from -360 to 0 degrees.

The Bode diagram (logarithmic amplitude characteristic, LAH), which is defined as  $L = 20 \lg A(\omega)$  is measured in decibels and plotted as a function of  $\lg \omega$ . The LAC determines the change in the logarithm of the frequency function modulus when the frequency changes.

Logarithmic frequency characteristics are used in the analysis of such properties of systems as phase and amplitude (modulus) margins, transmission coefficient, bandwidth, and system response to disturbances.

The phase margin is the difference between the value of the phase frequency response and -180 degrees at the cutoff frequency.

The cutoff frequency is a frequency where the value of the amplitude frequency response is 1.0, or 0 dB.

The modulus margin is the value of the amplitude frequency response at a frequency where the phase frequency response is -180 degrees.

Typically, a phase margin between 30 and 60 degrees provides an acceptable compromise between stability and bandwidth.

Decibels are used as a unit of measurement for the value of  $20 \lg A(\omega)$ :

1 dB = 0.1 bel.

1B corresponds to a 10-fold amplification of the signal power, 2B corresponds to a 100-fold amplification, 3B corresponds to a 1000-fold amplification, and so on.

Since the power of the signal  $P$  is proportional to the square of the amplitude,  $\lg A^2 = 2 \lg A$ , the gain in bels, expressed in terms of the ratio of amplitudes  $A$ , is equal to  $2 \lg A$  (respectively, in decibels it is equal to  $20 \lg A$ ). In this case, the values of  $A(\omega)$  and  $L(\omega)$  are related to each other by the following ratios:

the "+" sign  $L(\omega)$  corresponds to gain and  $A(\omega) > 1$ ,  $L(\omega) > 0$ ;

the sign "-"  $L(\omega)$  corresponds to attenuation and  $A(\omega) < 1$ ,  $L(\omega) < 0$ .

|             |       |      |     |       |      |     |                        |      |    |     |      |
|-------------|-------|------|-----|-------|------|-----|------------------------|------|----|-----|------|
| A (oh) .... | 0,001 | 0,01 | 0,1 | 0,316 | 0,89 | 1,0 | 1,12 = $\sqrt[20]{10}$ | 3,16 | 10 | 100 | 1000 |
| L, dB ....  | -60   | -40  | -20 | -10   | -1   | 0   | 1                      | 10   | 20 | 40  | 60   |

With the use of LAC, the range of change in the gain is significantly narrowed.

The LPH is constructed in semilogarithmic coordinates, that is, as a dependence  $\psi(\omega)$  on  $\lg \omega$ ; the use of a logarithmic scale on the ordinate axis does not make sense, since the phase shift of the chain of links (multiplied) is obtained as the sum of the phase shifts of individual links.

The unit of increment  $\lg \omega$  is a decade corresponding to a 10-fold change in frequency, or an octave corresponding to a two-fold change in frequency (1 octave is equal to 0.303 decans, since  $\lg 2 = 0.303$ ). The inclination of the LAC corresponds to 20 dB/dec  $\cong$  of 6 dB/oct (6.06 dB/oct to be exact).

### Sequence of work

To perform practical work, the Control System Toolbox application package (PPP) is used, which is designed to work with LTI (Linear Time Invariant Models) models of control systems.

Control System Toolbox has a data type that defines a dynamic system as a complex transfer function. The command syntax that creates an LTI system with one input and one output as a transfer function:

$\text{TF}([b_m, \dots, b_1, b_0], [a_n, \dots, a_1, a_0])$

$b_m, \dots, b_1$  are the values of the coefficients of polynomial  $B$  (1.3),

$a_n, \dots, a_1$  are the values of the coefficients of polynomial  $A$  in (1.3).

To perform the work, the commands given in Table 1.1 can be used.

Table 1.1. - Some Control System Toolbox Commands

| Syntax                                    | Description                                                                                                     |
|-------------------------------------------|-----------------------------------------------------------------------------------------------------------------|
| <code>pole(&lt;LTI object&gt;)</code>     | Calculating the Poles of the Transfer Function                                                                  |
| <code>zero(&lt;LTI-объект&gt;)</code>     | Calculation of Zeros of the Transfer Function                                                                   |
| <code>K=dcgain(&lt;LTI-объект&gt;)</code> | Calculates the transfer factor $k$                                                                              |
| <code>step(&lt;LTI-model&gt;)</code>      | Construction of a transient process graph. Calculates and plots the model's response to a single step function. |
| <code>impulse(&lt;LTI-объект&gt;)</code>  | Plotting the Pulse Transient Function                                                                           |

|                       |                                                                                                                                                          |
|-----------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------|
| bode(<LTI object>)    | Construction of logarithmic amplitude-frequency response and phase-frequency response (Bode diagrams).                                                   |
| nyquist(<LTI-объект>) | Construction of the Nyquist frequency hodograph                                                                                                          |
| Nichols(<LTI-объект>) | Construction of the Nichols frequency hodograph                                                                                                          |
| ngrid                 | Plots a diagram (grid) on the Nichols frequency hodograph                                                                                                |
| margin                | Calculates stability margins by phase and amplitude                                                                                                      |
| norm                  | Calculates the maximum amplitude value and the corresponding frequency value                                                                             |
| zgrid                 | Mesh constant damping factor lines from 0 to 1 in 0.1 increments and natural frequencies from 0 to 10 rad/s in 1 rad/s increments on the complex plane z |
| evalfr(sys,f)         | Calculates the value of the complex transmission ratio for one given complex frequency value                                                             |

To determine the roots of degree k polynomials, the MATLAB **roots(P) command can be used**, which, as an argument P, receives a matrix of polynomial coefficients  $[pk, \dots, p0]$ .

Another option for obtaining graphs of the dynamic characteristics of ACS is to use the CST – LTI viewer graphical interface, which is called by the **ltiview** command, which has the name of the variable containing the LTI object as a parameter.

As an example, let's look at the use of the basic commands of the Control System Toolbox.

### Methodical example

The transfer function of the ACS is specified

$$W = \frac{s + 2}{3s^3 + 4s^2 + 5s + 3}.$$

We will find its dynamic and frequency characteristics. We will work in the command mode of the MATLAB environment.

1. Create an LTI object named w, to do this, we will run:

```
>> w=tf([1 2],[3 4 5 3])
```

Transfer function:

$$s + 2$$

-----  
 $3 s^3 + 4 s^2 + 5 s + 3$

2. Determine the location of the poles and zeros for a given transfer function on the complex plane using the pzmap (w) command in Fig. 3.

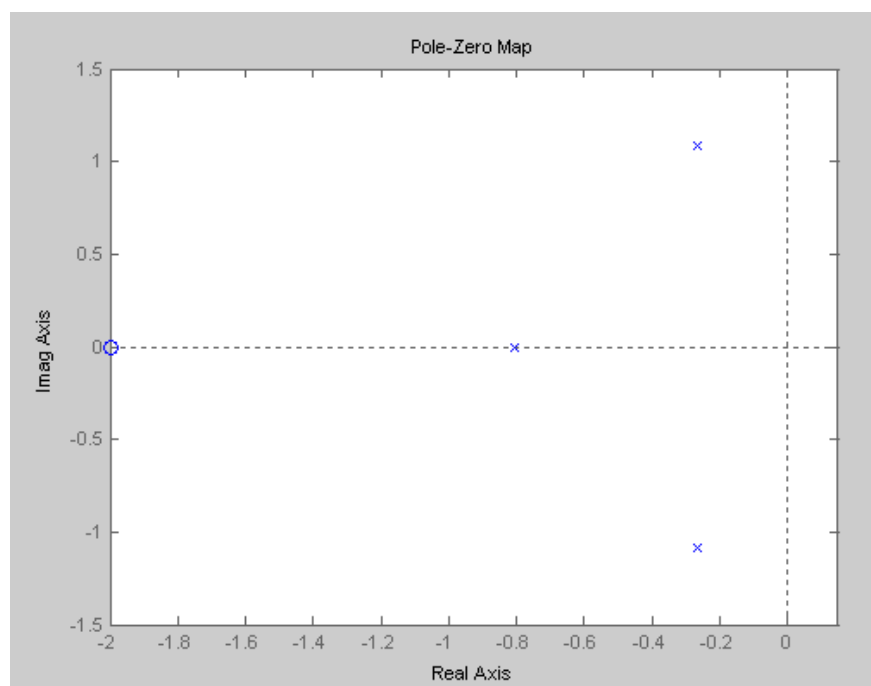


Figure 3. – Location of poles and units for a given transfer function

3. Mesh the damping coefficient and frequency constant lines on the z-plane using the zgrid command in Fig. 4.

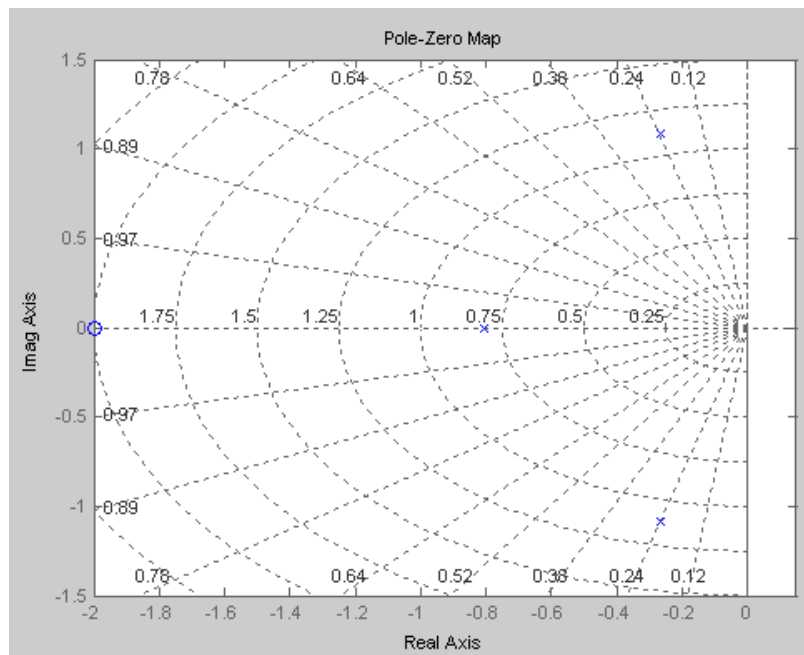


Figure 4. – Location of poles and zeros on the grid of constant lines of the damping coefficient and frequency on the z plane

4. Calculate the complex transmission value for a given frequency value using the `evalfr(w,2)` command

```
>> evalfr(w,2)
```

```
years = 0.0755
```

In this example, the frequency is 2 rad/s.

5. Calculate the transfer ratio using the command `k=dcgain(w)`

```
>> k=dcgain(w)
```

```
k = 0.6667
```

6. Obtain the logarithmic frequency response using the `bode(w)` command – Fig. 5.

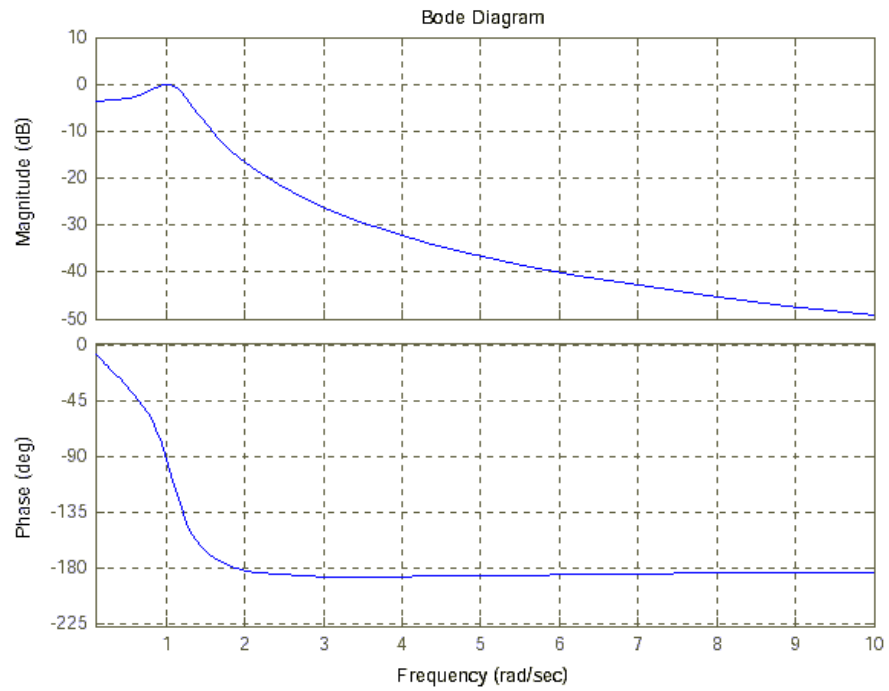


Figure 5. - Logarithmic frequency response of the system

6. Determine the frequency Nyquist hodograph by running the command `nyquist(w)` – Fig. 6.

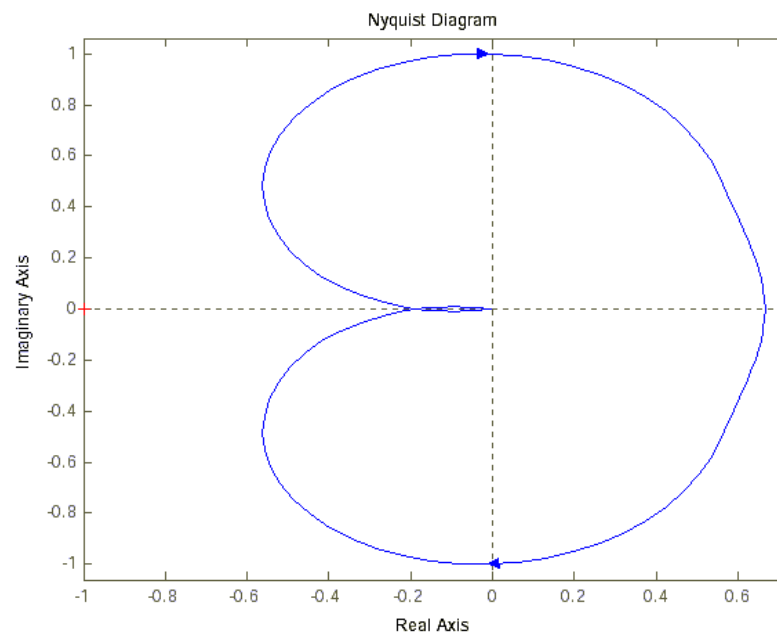


Figure 6. - Frequency hodograph

7. Determine the safety margin of the system by phase and modulus by running the command `margin(w)` of Fig. 8.

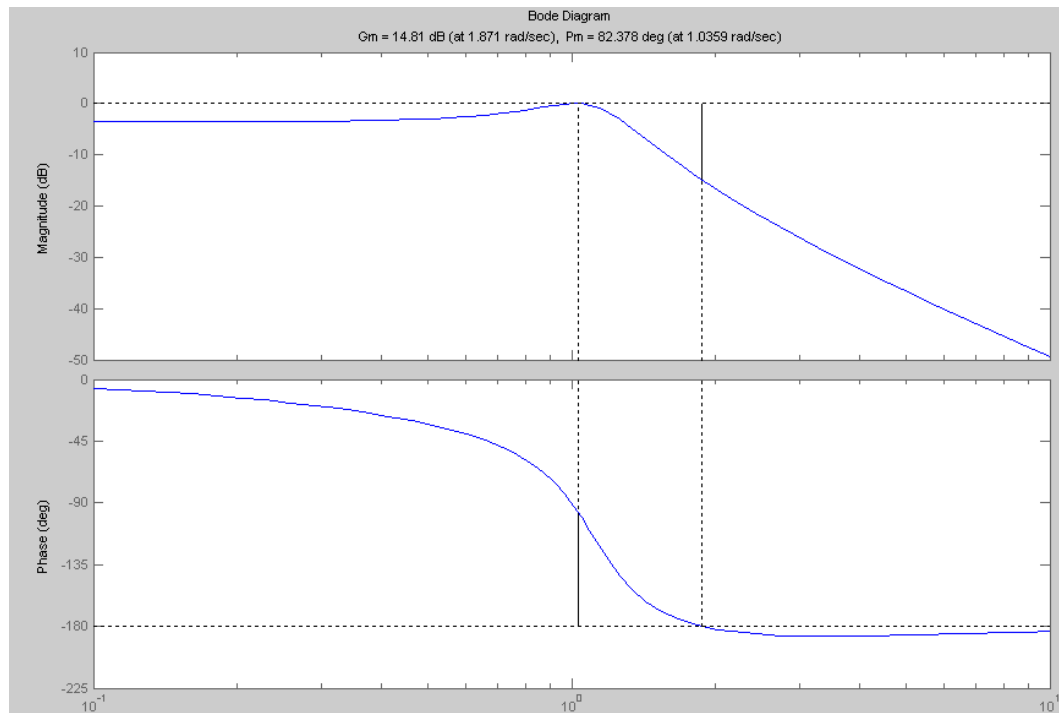


Figure 8. – Graphs for determining the safety margin of the system by phase and modulus

8. Let's calculate the maximum amplitude value and the corresponding frequency value:

```
>> [peak,fprak]=norm(w,inf)
```

```
peak = 0.9958
```

```
fprak = 1.0498
```

Let us construct a Nichols hodograph by executing the command `nichols(w)` of Fig. 9.

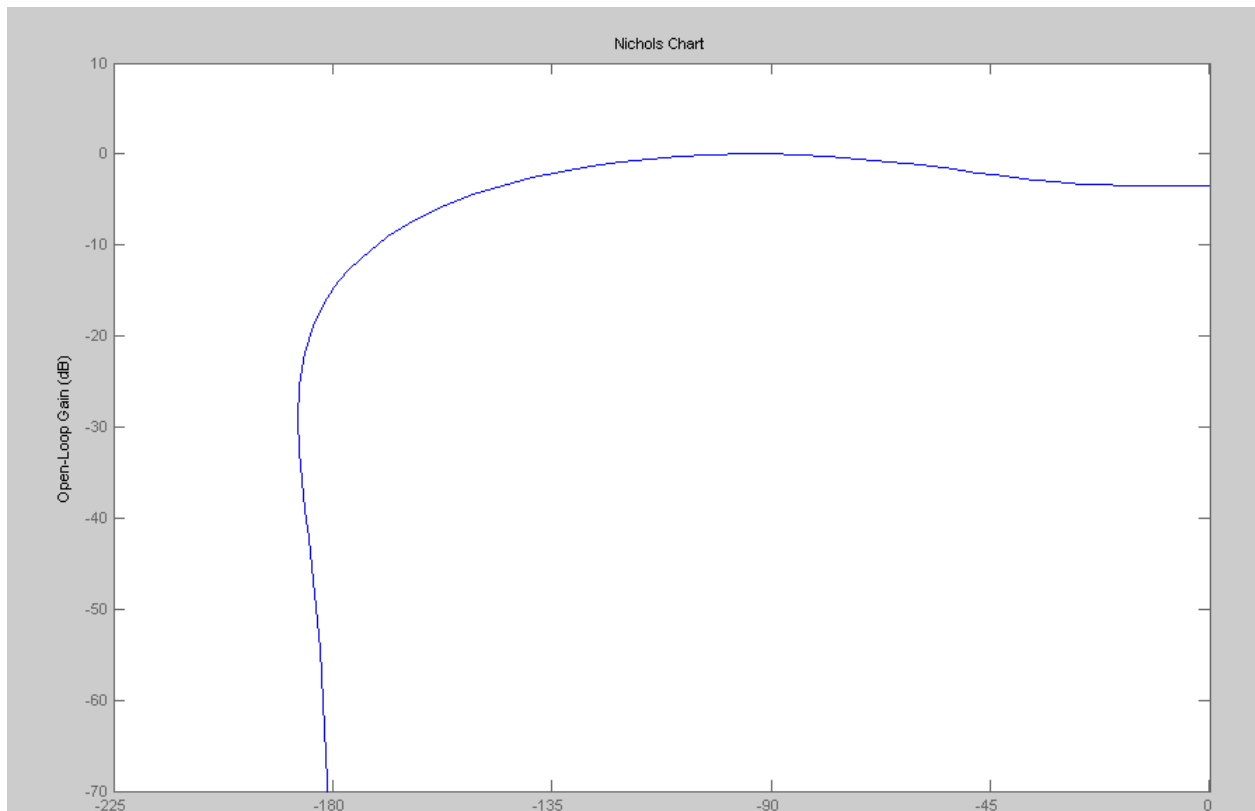


Figure 9. Nichols' hodograph

9. Let's plot a Nichols chart (grid) by entering the ngrid command, Fig. 10.

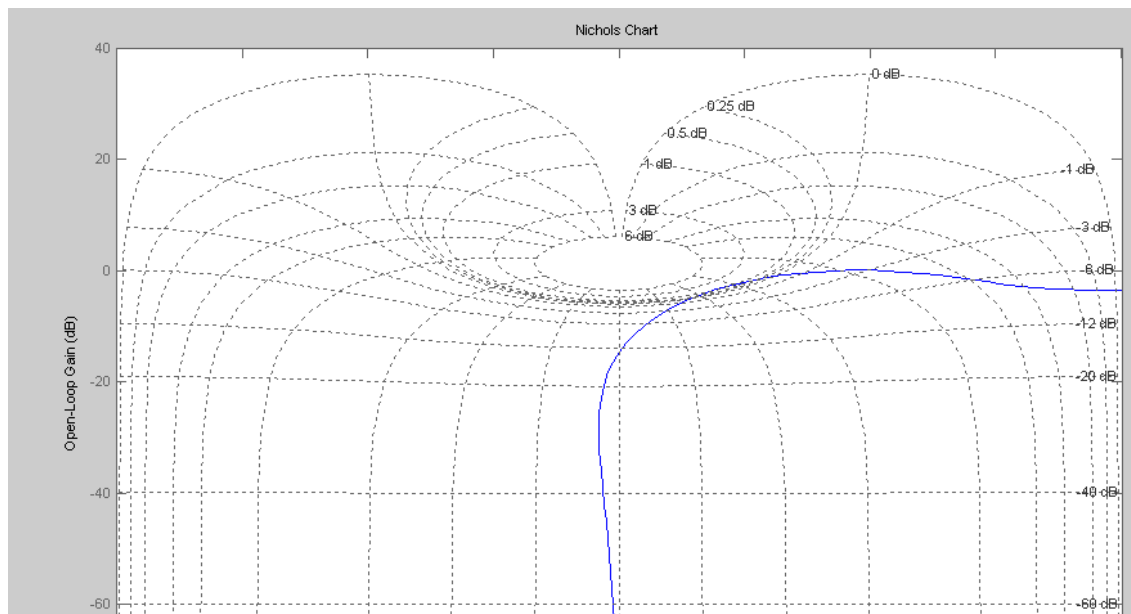


Figure 10. – Nichols diagram

Using the Nichols hodograph and the Nichols diagram superimposed on it, the properties of both the open and closed models are analyzed. The ngrid command plots the constant amplitude and phase lines corresponding to the following transformation of the complex plane  $H$ :

$$W = \frac{H}{1+H}.$$

Such a coordinate grid is called a Nichols diagram. For one-dimensional models, it connects the points of the Nichols hodograph for an open system with the points of the hodograph for a system closed by a single negative feedback. The ngrid command builds a grid for a module in the range from -40 to 40 dB, for a phase - from -360 to 0 degrees.

Each of the constructed characteristics fully and unambiguously determines the management system under consideration.

## Variants of tasks

| №  | Type of transfer function                                                    | №  | <i>Coefficients of polynomials</i> |       |       |       |       |       |       |
|----|------------------------------------------------------------------------------|----|------------------------------------|-------|-------|-------|-------|-------|-------|
|    |                                                                              |    | $b_0$                              | $b_1$ | $a_0$ | $a_1$ | $a_2$ | $a_3$ | $A_4$ |
| 1. | $W(p) = \frac{b_1 p + b_0}{a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0}$       | 1. | 0                                  | 3     | 1     | 2     | 3     | 0     | 1     |
|    |                                                                              | 2. | 2                                  | 6     | 4     | 0     | 1     | 5     | 1     |
|    |                                                                              | 3. | 0                                  | -3    | 5     | 2     | 0     | 2     | 1     |
|    |                                                                              | 4. | 4                                  | 2     | 3     | 4     | 5     | 3     | 1     |
|    |                                                                              | 5. | 0                                  | 1     | -2    | -2    | -3    | -2    | 0     |
|    |                                                                              |    | $b_0$                              | $b_1$ | $b_2$ | $a_0$ | $a_1$ | $a_2$ | $A_3$ |
| 2. | $W(p) = \frac{b_2 p^2 + b_1 p + b_0}{a_3 p^3 + a_2 p^2 + a_1 p + a_0}$       | 1. | 0                                  | -3    | 2     | 4     | 2     | 3     | 9     |
|    |                                                                              | 2. | 8                                  | 0     | -3    | -4    | -6    | -4    | -1    |
|    |                                                                              | 3. | -4                                 | 6     | -2    | 5     | 5     | 0     | 1     |
|    |                                                                              | 4. | 6                                  | -8    | -7    | 0     | -6    | -3    | -1    |
|    |                                                                              | 5. | 2                                  | -1    | -3    | -1    | 0     | -7    | -2    |
|    |                                                                              |    | $b_0$                              | $b_1$ | $b_2$ | $a_0$ | $a_1$ | $a_3$ | $a_4$ |
| 3. | $W(p) = \frac{b_2 p^2 + b_1 p + b_0}{a_4 p^4 + a_3 p^3 + a_1 p + a_0}$       | 1. | 0                                  | 2     | 8     | -3    | 7     | -7    | 1     |
|    |                                                                              | 2. | -5                                 | 0     | 3     | -8    | -2    | -1    | -6    |
|    |                                                                              | 3. | -7                                 | 1     | 2     | 0     | 5     | 2     | 9     |
|    |                                                                              | 4. | -6                                 | 4     | -4    | 1     | 0     | 6     | 3     |
|    |                                                                              | 5. | 2                                  | -2    | -1    | 5     | 3     | 0     | 9     |
| 4. | $W(p) = \frac{b_2 p^2 + b_0}{a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0}$     | 1. | 0                                  | -5    | 4     | 3     | 7     | 9     | 1     |
|    |                                                                              | 2. | 7                                  | -6    | 0     | 5     | 8     | 2     | 2     |
|    |                                                                              | 3. | -2                                 | -8    | 2     | 0     | 4     | 3     | 3     |
|    |                                                                              | 4. | -7                                 | -1    | 6     | 9     | 0     | 4     | 2     |
|    |                                                                              | 5. | -3                                 | 7     | -4    | 4     | 5     | 0     | 1     |
|    |                                                                              |    | $b_2$                              | $b_3$ | $a_0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
| 5. | $W(p) = \frac{b_3 p^3 + b_2 p^2}{a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0}$ | 1. | 0                                  | -5    | 4     | 3     | 7     | 9     | 1     |
|    |                                                                              | 2. | 7                                  | -6    | 0     | 5     | 8     | 2     | 2     |
|    |                                                                              | 3. | -2                                 | -8    | 2     | 0     | 4     | 3     | 3     |
|    |                                                                              | 4. | -7                                 | -1    | 6     | 9     | 0     | 4     | 2     |
|    |                                                                              | 5. | -3                                 | 7     | -4    | 4     | 5     | 0     | 1     |

## Procedure for performing practical work

1. To study theoretical information.
2. Start the MATLAB system.
3. Create a tf object according to the specified option.
4. Determine the poles of the transfer function  $s_j^*, (i = \overline{1, n})$  using the roots or pole command.
5. Determine the zeros of the transfer function  $s_j^0, (j = \overline{1, m})$  using the roots or zero command.
6. On the complex plane, obtain the location of poles and zeros. From the location of the roots, draw a conclusion about the stability of the ACS under study.
9. Calculate the transfer ratio using the command  $k=dcgain(w)$ .
10. Calculate the complex transmission value using the evalfr(w,f) command for frequencies 1, 10, 100, 1000 rad/s. Determine the change in the complex transmission value when the frequency changes. If the complex transmission value changes, determine its maximum and minimum values for the frequency range from 1 to 10000 rad/s. Draw conclusions from the result.
11. Mesh the constant lines of the damping coefficient and frequency on the complex z-plane.
12. Using the LTI-viewer, or the corresponding commands (Table 1), obtain frequency characteristics – Bode diagram, Nyquist frequency hodograph. Analyze the results obtained for each studied characteristic and draw quantitative and qualitative conclusions.
13. Determine the stability margin of the system under study by phase and modulus.
14. Calculate the maximum amplitude value and the corresponding frequency value.
15. Construct a Nichols hodograph and plot a Nichols diagram on it.
16. Draw conclusions based on the results obtained.
17. Answer control questions.
18. Draw up a report.

19. Submit the report to the teacher and defend the work.

The report is drawn up in accordance with the requirements for the design of laboratory work at the university, and must contain a title page, a statement of the purpose of the work, a statement of the task in accordance with the version of the task, the results of the work, and conclusions.

Note: Variants of tasks consist of two digits: the first is the number of the transfer function, the second is the number of the set of coefficient values.

### **Control questions**

1. Define the complex transfer function of the system.
  2. Define the frequency response of a stationary dynamic system.
  3. What is the relationship between the complex transfer function and the frequency response of the system? Write down expressions for amplitude-frequency, phase-frequency, real-frequency, imaginary frequency responses.
  4. Define the amplitude-phase response or Nyquist frequency hodograph.
  5. How are logarithmic frequency responses determined and what are they used for?
  6. What is the frequency hodograph and Nichols diagram used for?
12. How is the safety margin of the system determined by phase and modulus?

## PRACTICAL WORK

### Quality Assessment of Transient Processes of Linear Systems

Study questions:

1. Study of the quality of transient processes using direct indicators.
2. Study of the quality of transient processes using indirect indicators.

#### Brief theory

##### Static and dynamic models

Often the transient processes in the designed system and its static states are studied separately. In this case, to analyze one object, it is necessary to build at least two models: static and dynamic.

A static model is, as a rule, a system of Boolean, algebraic or transcendental equations, the solution of which determines the state of the object under study. The main feature of such models is the absence of an independent variable - time in their description.

$$\bar{Y} = W_c(\bar{I}, \bar{X}, \bar{Q}) \text{ или } \bar{Y} = W_c(\bar{X}, \bar{Q}) \text{ или } \bar{Y} = W_c(\bar{Q})$$

There is no time model time in the above functions.

With the help of static models, it is possible to assess the static accuracy of the object, study its static errors and build the main static characteristics: input, transfer and output.

Dynamic models simulate the behavior of an object in time. They have an explicit or implicit presence of time:

$$\bar{Y} = W_c(\bar{I}, \bar{X}, \bar{Q}, t_m) \text{ или } \bar{Y} = W_c(\bar{Q}, t_m)$$

With the help of dynamic models, transient processes in the system are investigated, the speed and delays of signal propagation through the elements of the system are determined, and transient characteristics are constructed.

Dynamic models of continuous systems are systems of differential equations. For discrete systems, it is necessary to solve the main question: how to organize a "model clock", that is, to choose a specific mechanism for advancing model time.

At the end of the transient processes, the system reaches a steady state. This indicates the possibility of verification (verification) of static and dynamic models by the coincidence of the results of "tests" of static models with the results obtained on dynamic models.

The processes occurring in discrete dynamical systems can be represented by a chain of transitions from one state to another at discrete moments in time. This provides another opportunity to study dynamical systems on a certain set of static models, and to simulate time in an implicit way, for example, by linking it to the cycle of operation of the control modeling program.

### **Systems in industrial plants**

In industrial installations with automatic control systems, two groups can be distinguished:

1. Plants in which the transient time is negligible compared to the steady state. In these cases, dynamic modes do not have a significant impact on product quality and performance, e.g. pumps, fans, conveyors, etc.
2. Installations in which the transient time is commensurate with the steady-state time, or in which the deviation of the variable in dynamics has a significant impact on product quality, e.g. machine tools, robots, tracking systems, etc.

Stability, which is characterized by the attenuation of transient processes, is a necessary, but far from sufficient condition for the practical suitability of systems. This criterion allows a rather rough assessment of transient processes. Let us consider what the transient characteristics of systems look like for the three main modes of operation from the point of view of stability in Fig.1.

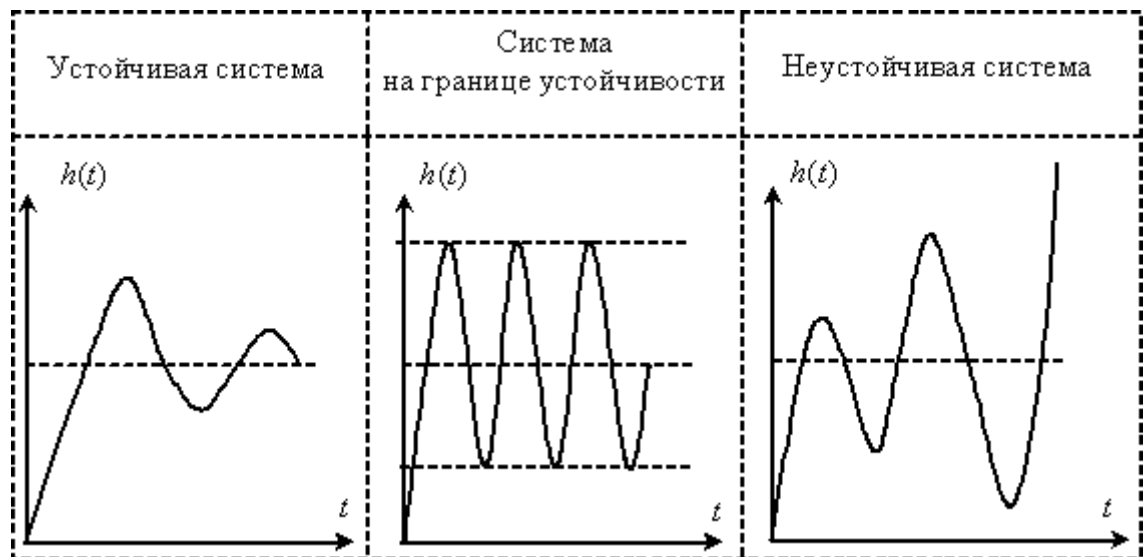


Figure 1 – Transients of systems

Using, as shown in Fig. 1, the criterion of stability according to the type of transient characteristic (transient process), it can be concluded that the system is stable if the transient process is dampened. However, the system can be stable, but its transient processes, depending on the change in parameters, will differ significantly in Fig. 2.



Figure 2 – Transients of a sustainable system

It is obvious that these processes differ significantly in appearance, have different frequency of oscillations, completion time, and amplitude of deviation from a given value. In this regard, there is a need to compare and evaluate the stable time characteristics of the systems.

It should be noted that specialists are interested not only in transient processes when changing control actions, but also in transient processes when perturbing actions change.

Thus, the concept of the quality of transient processes **is important for control systems**. For practice, it is important to assess the nature of the processes, especially such factors as duration, oscillation and dynamic deviation of output characteristics from a given value.

To assess the quality of transients, characteristics, criteria or quality indicators are required, which can be expressed numerically.

Quality criteria are used when it is necessary to carry out:

1. Comparative analysis of automatic control systems when changing the parameters of the control object, or when comparing systems of different types for the same control object.
2. Selection of parameters (synthesis) of automatic control systems that provide the specified criteria for the quality of transient processes, the requirements of the technical specification for the development of the system.

It is known that the transient process in the control system depends not only on the properties of the system itself, but also on the nature (type) of the input impact. Therefore, the behavior of the system in their assessment is considered under typical external influences. As such typical influences, the following are most often used:

- a single step function, less often, a linearly rising signal,
- influence of the harmonic function.

Quality assessments are divided into two groups:

1. **Direct indicators** of the quality of transient processes. They directly characterize the transition process itself, the reaction of the system to a typical impact, most often, to a single step function.
2. **Indirect quality indicators**. They evaluate the quality of transient processes by other characteristics of the system, such as frequency characteristics, the nature and location of the roots of the characteristic equation (poles of the transfer function), integrals of the temporal function of the transient process.

### **Evaluation of the quality of the transient process under the influence of a step function**

Direct quality estimates are determined according to the graph of the transient characteristic of the control system  $h(t)$ , that is, when the system is affected by a single step function

$$1(t) = \begin{cases} 1 \leftarrow t > 0, \\ 0 \leftarrow t \leq 0, \end{cases}$$

at zero initial conditions, or along the transient curve of the adjustable variable  $X(t)$  when the input is acted upon by a step function with an amplitude corresponding to the nominal or other defined value of the controlled variable.

Let us consider a control system with a single negative feedback, the structural diagram of which is presented in Fig. 3.

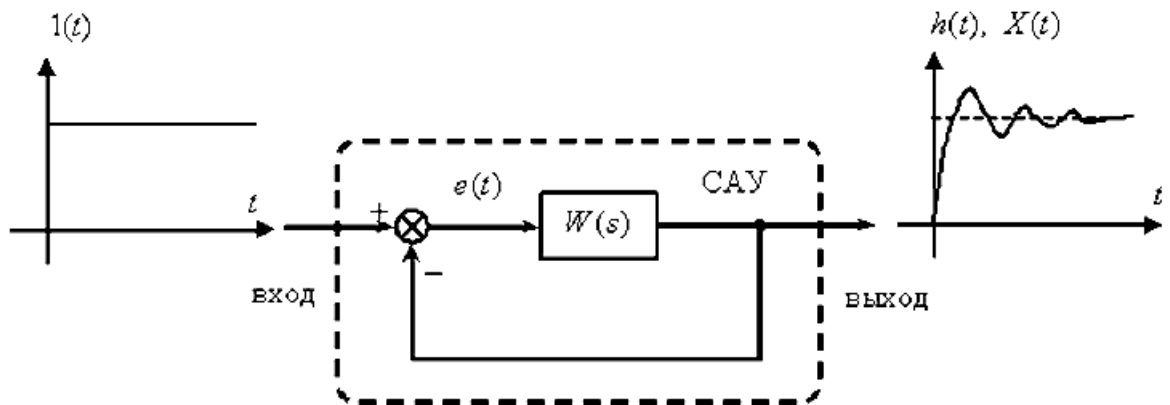


Figure 3 – Control system with single negative feedback

A step signal is received at the input of the system, and the response of the  $h(t)$  system can be observed at the output of the system. In addition, a change in the control error can be considered as a transient process

$$e(t) = X_{\text{зад}}(t) - X(t),$$

when the desired output characteristic  $X_{\text{зад}}(t)$  is determined.

Approximate graphs of the signal change at the output and the throttling error are shown in Figs. 4 and 5, respectively.

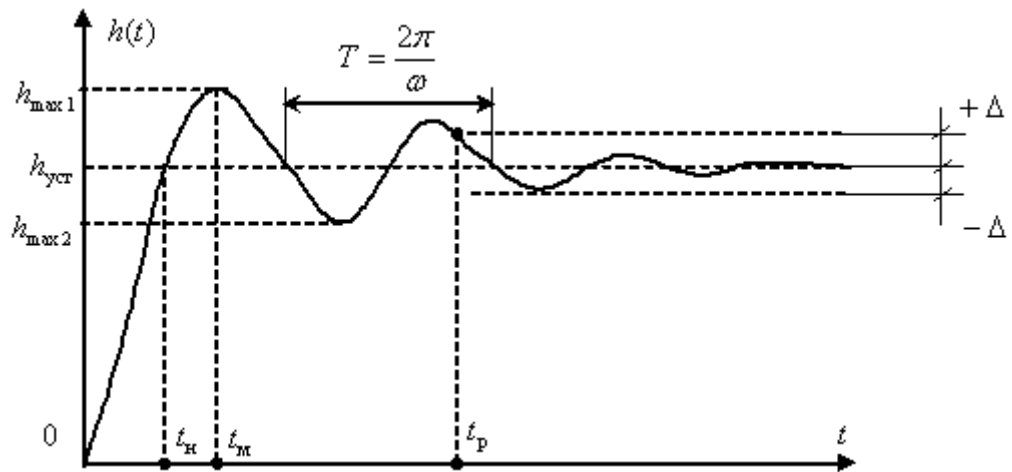


Figure 4 – Output signal change graph

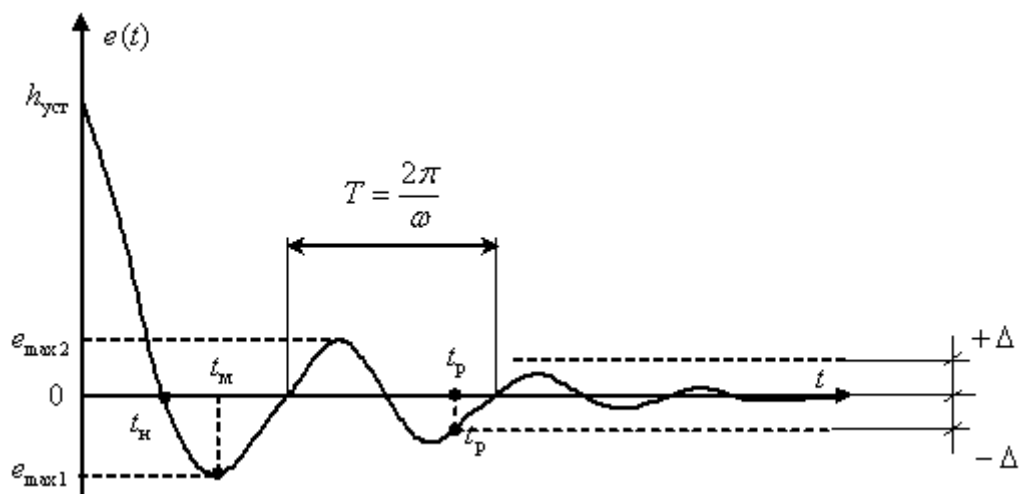


Figure 5 – Graph of the change in the error signal

Let us consider the direct estimates of the quality of transient processes shown in Fig. 4 and 5:

1.  $t_p$  – the minimum time after which the regulated value will remain close to the steady-state value with a given accuracy

$$|h(t) - h_{yct}| \leq \Delta, \quad \Delta = \text{const.}$$

$\Delta$  is pre-set as a percentage of the steady-state value  $h_{yct}$ , where there are no specific requirements, they accept  $\Delta\% = 5\%$ .

2.  $\sigma$  – overregulation – the maximum deviation from the established value, expressed in relative units or percentages

$$\sigma = \frac{h_{\max 1} - h_{yct}}{h_{yct}} 100\% \text{ or } \sigma = \frac{|e_{\max 1}|}{e(0)} 100\% = \frac{|e_{\max 1}|}{h_{yct}} 100\%.$$

Usually, the requirements for overregulation are  $\sigma \approx 10 \div 30\%$ . Sometimes there may be a requirement for the quality of transients  $\sigma = 0\%$ , for example, in positioning systems for industrial robot arms.

3.  $\omega$  – frequency of oscillations

$$\omega = \frac{2\pi}{T},$$

where  $T$  – oscillation period for oscillatory processes.

4.  $N$  is the number of complete oscillations that  $h(t)$  or  $e(t)$  during the regulation period  $t_p$ , Normal Fluctuation Number Requirements  $N \approx 1 \div 2$ , some systems impose a limit on oscillation  $N = 0$ , for example, in systems with significant backlash in mechanical transmissions.

5.  $t_m$  – the time of reaching the first maximum.

6.  $t_{\kappa}$  is the time of the transient process increase, the time from the beginning of the transient process to the moment of the first crossing of the steady-state value line by the graph.

7.  $\kappa$  is an attenuation decrement equal to the ratio of the modules of two adjacent readjustments

$$\kappa = \frac{|h_{\max 1} - h_{yct}|}{|h_{\max 2} - h_{yct}|}.$$

The above indicators can be supplemented by others, if this is required by specific technical specifications for the development or study of control systems.

Stable transient processes arising from step-by-step action are usually divided into three groups:

1. Monotonic processes. These are processes where the first derivative of the output quantity does not change sign over time.
2. Aperiodic processes. Here the derivative changes sign no more than once.
3. Oscillatory processes. The derivative changes its sign periodically.

Fig. 6 shows an approximate type of oscillatory, aperiodic and monotonic processes.

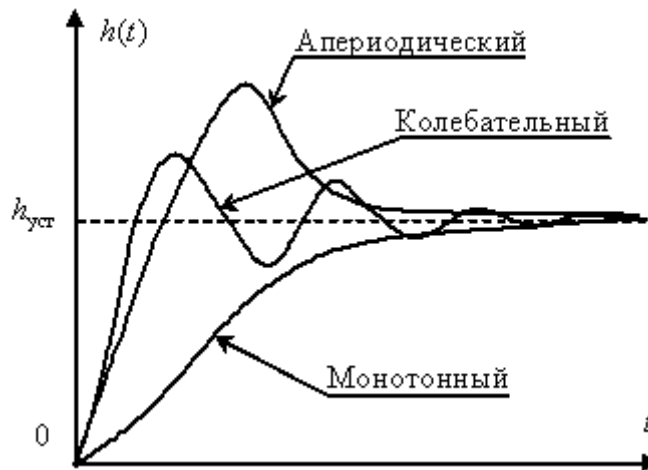


Figure 6 – Approximate type of oscillatory, aperiodic and monotonic processes

The requirements for the quality of the transient process can be represented graphically. They boil down to the requirement that the deviation of the controlled value under the stepped action does not go beyond the boundaries of a certain area depicted in the process quality diagram. The simplest diagram of the quality of the process under study is shown in Fig. 7.

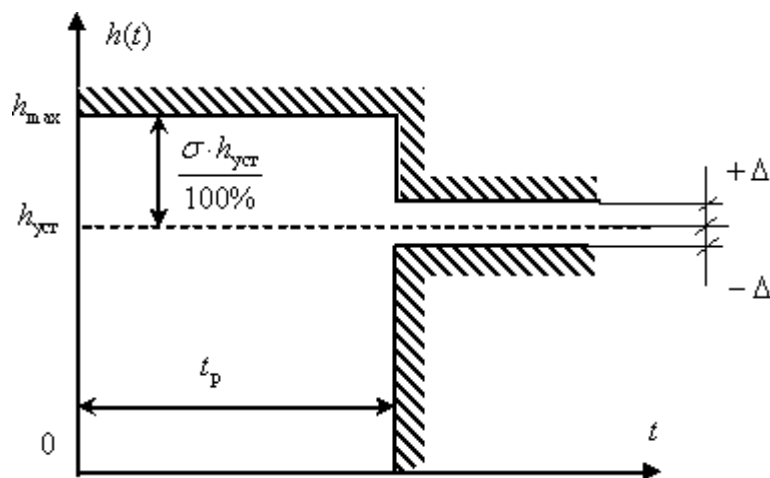


Figure 7 – The simplest process quality diagram

The refined diagram, in which restrictions are imposed on the rate of slew of the signal, is shown in Fig. 8. On the refined diagram  $t_s$  denotes the lag time equal to the time interval between the moments of application of the jump signal ( $t = 0$ ) and the point in time at which the output reaches half of the steady-state value.

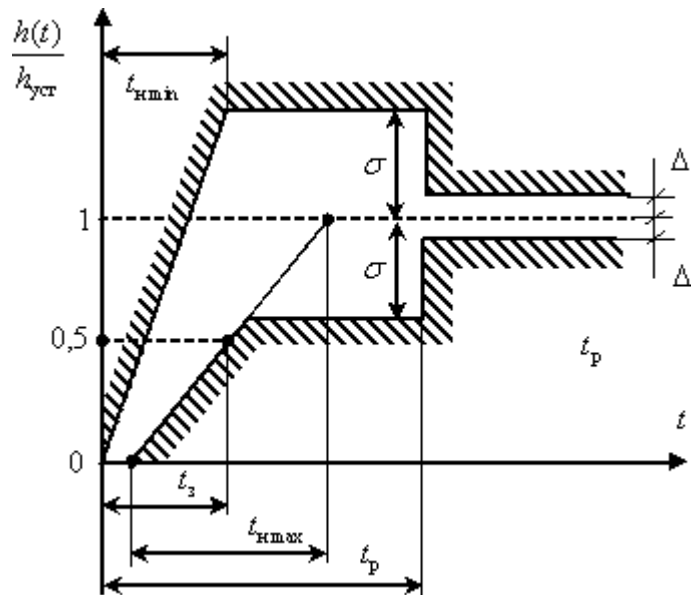


Figure 8 – Refined process quality diagram

Methodical example:

The transfer function of the ACS is specified:

$$W := \frac{(2s + 3)}{(4s^3 + 5s^2 + 6s + 4)}$$

1. Create an LTI object named w, to do this, we will run:

```
>> w=tf([2 3],[4 5 6 4])
```

Transfer function:

2 s + 3

-----  
4 s^3 + 5 s^2 + 6 s + 4

2. Define the transient function using the step(w) command. The result of this command is shown in Fig. 9.

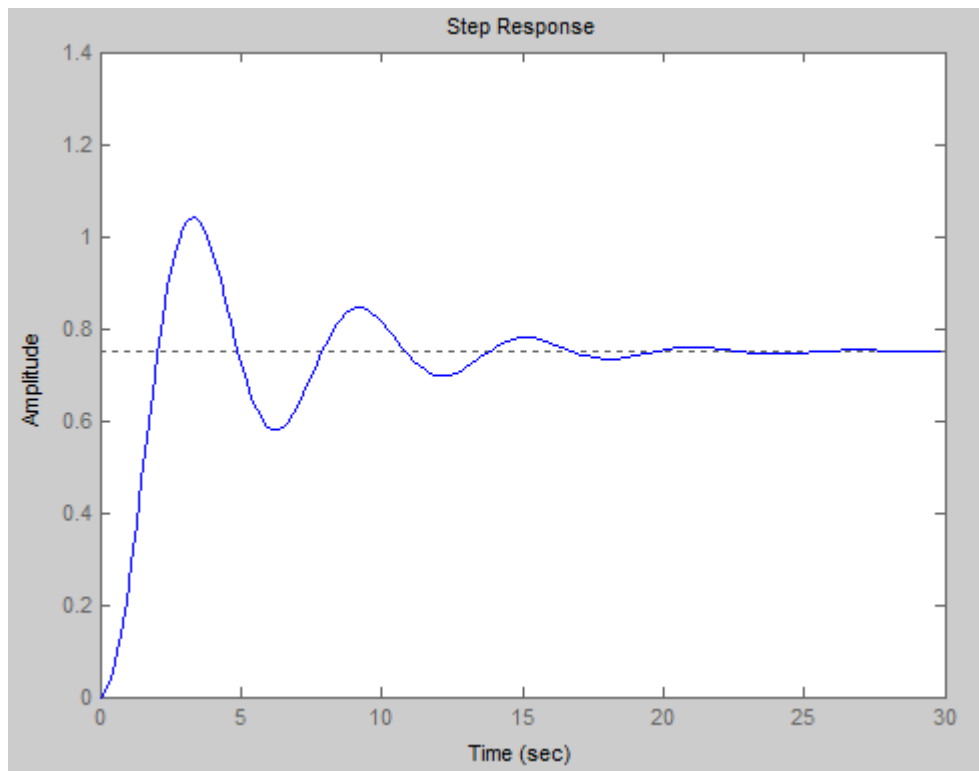


Figure 9 - Transient function  $h(t)$

3. Let's calculate direct estimates of the quality of transition processes.

1.  $t_p$  is the minimum time after which the regulated value will remain close to the steady-state value:

TP – 10.2 SEK with a given accuracy –  $|h(t) - h_{yct}| \leq \Delta$ ,  $\Delta = const$ ;

$$|0.8 - 0.75| \leq \Delta; |0.05| \leq 5.$$

$\Delta$  is preliminarily set as a percentage of the established value of the HUST, where there are no specific requirements –  $\Delta = 5\%$  is taken.

2.  $\sigma$  – overshooting – maximum deviation from the established value:

$$\sigma = \frac{h_{\max} - h_{yct}}{h_{yct}} 100\% \text{ or } \sigma = \frac{|e_{\max}|}{e(0)} 100\% = \frac{|e_{\max}|}{h_{yct}} 100\%.$$

$$\sigma = \frac{(1.05 - 0.75)}{0.75} * 100\% = 40\%$$

3.  $\omega$  – frequency of oscillations –

$$\omega = 2\pi/T,$$

where T is the oscillation period for oscillatory processes. Determine the oscillation period from the graph – 6 seconds. Calculate the oscillation frequency:

$$\omega = 2 * 3.14 / 6 = 1.05$$

4. Determine the number of complete oscillations N during the regulation of tp.

$$N=1.$$

5. Determine tm – the time to reach the first maximum:

$$t_m = 3.3 \text{ sec.}$$

6. Determine tn is the time of the transient process increase:

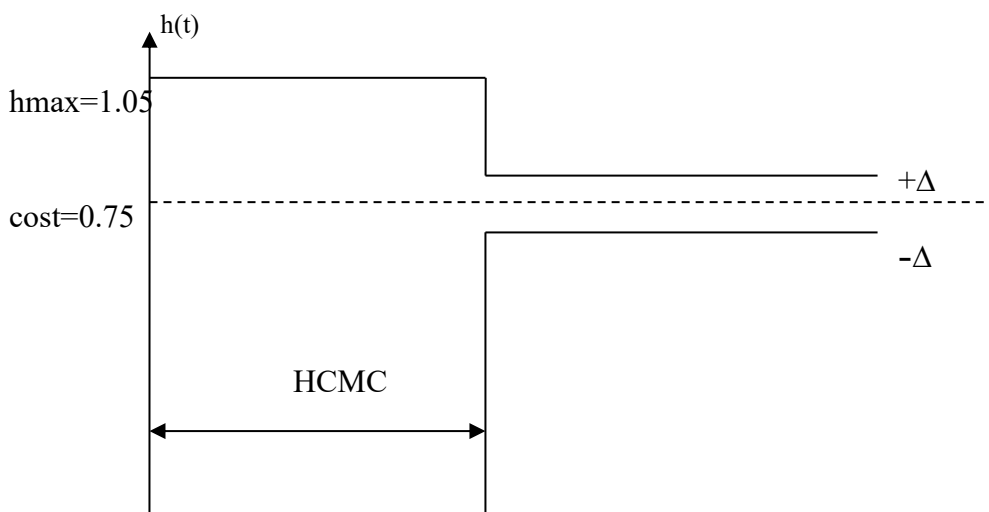
$$t_n = 2.1 \text{ sec.}$$

7. Calculate k is the decrement of attenuation:

$$K = \frac{|h_{\max 1} - h_{yct}|}{|h_{\max 2} - h_{yct}|}.$$

$$k = |1.05 - 0.75| / |0.58 - 0.75| = |0.3| / |0.17| = |1.76|$$

4. Let us define the requirements for the quality of the transient process graphically. They boil down to the requirement that the deviation of the controlled value in the case of a stepwise action should not go beyond the boundaries of a certain area depicted in the process quality diagram.



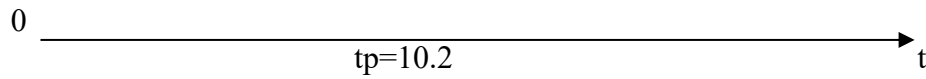


Figure 10 – Process quality diagram

5. Let's build a refined diagram in which restrictions are imposed on the rate of signal rise.

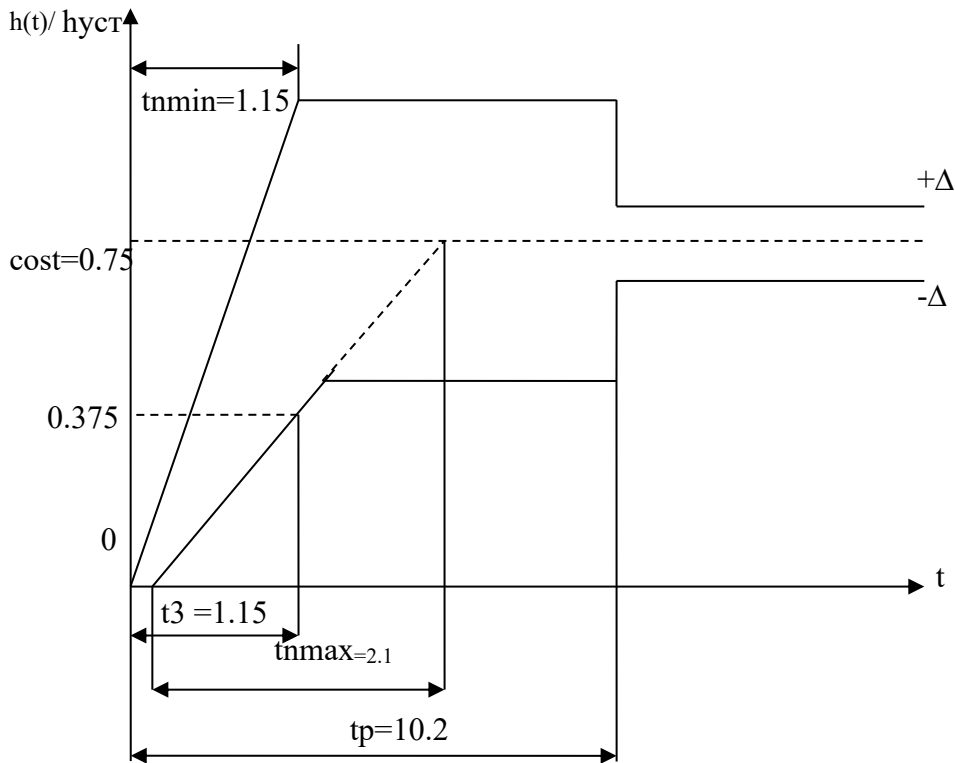


Figure 11 – Refined diagram

### Variants of tasks:

The automatic control system is set by the transfer function

$$W := \frac{K}{T^2 S^2 + 2\xi TS + 1}$$

| Variant number | K Value | T Value | The meaning of $\xi$ |
|----------------|---------|---------|----------------------|
| 1              | 3       | 2       | 0.1                  |
| 2              | 1       | 1       | 0.4                  |
| 3              | 6       | 2       | 0.5                  |
| 4              | 4       | 1       | 0.1                  |
| 5              | 5       | 1       | 0.9                  |
| 6              | 2       | 2       | 0.7                  |

|    |     |     |     |
|----|-----|-----|-----|
| 7  | 7   | 6   | 0.2 |
| 8  | 2   | 3   | 0.1 |
| 9  | 0.4 | 1   | 0.5 |
| 10 | 7   | 2   | 0.4 |
| 11 | 0.8 | 4   | 0.3 |
| 12 | 5   | 0.7 | 0.1 |
| 13 | 8   | 0.2 | 0.5 |
| 14 | 0.9 | 4   | 0.4 |
| 15 | 1.8 | 7   | 0.7 |
| 16 | 2   | 2   | 0.4 |
| 17 | 3.8 | 3   | 0.9 |
| 18 | 5   | 1.8 | 0.8 |
| 19 | 6   | 4   | 0.6 |
| 20 | 8   | 0.1 | 0.7 |
| 21 | 9.9 | 2   | 0.1 |
| 22 | 5   | 7.2 | 0.2 |
| 23 | 3   | 3   | 0.3 |
| 24 | 1   | 6   | 0.3 |
| 25 | 2   | 4   | 0.6 |
| 26 | 5   | 1.5 | 0.8 |
| 27 | 4   | 5   | 0.5 |
| 28 | 8   | 3   | 0.6 |
| 29 | 8.1 | 7   | 0.9 |
| 30 | 10  | 8   | 0.1 |

### Procedure for performing practical work

1. Create an LTI object named  $w$ , for your variant.
2. Get a transient function using the  $\text{step}(w)$  command.
3. To calculate direct estimates of the quality of transient processes:
  - a)  $t_p$  is the minimum time after which the regulated value will remain close to the steady-state value;
  - b)  $\sigma$  – overregulation;
  - c)  $\omega$  is the frequency of oscillations;
  - d)  $N$  is the number of complete oscillations that  $h(t)$  has during the time of regulation  $t_p$ ;
  - e)  $t_m$  is the time to reach the first maximum;
  - f)  $t_n$  is the time of the transient process increase;

g)  $k$  is the decrement of attenuation;

4. To determine the requirements for the quality of the transient process graphically.
5. Build a refined diagram in which restrictions are imposed on the rate of signal rise.
6. To assess the quality of transient processes using indirect quality indicators: using frequency characteristics (use the Bode diagram to determine the stability margin by modulus and phase), to determine the degree of their remoteness from the stability boundary by the nature of the location of the roots of the characteristic equation (poles of the transfer function).
6. To draw conclusions about the work done.

### **Control questions:**

1. What is a static model? Describe it.
2. What characteristics of ACS are studied using dynamic models?
3. What is the main difference between static and dynamic models?
4. What is meant by the stability of systems?
5. Give the transient characteristics of the stability regimes: stable and unstable systems, systems on the stability boundary.
6. What is meant by the quality of the transition process?
7. What typical impacts are used to assess transients?
8. Describe the direct and indirect indicators of the quality of transient processes. What is their fundamental difference?
9. Give approximate graphs of monotonic, aperiodic and oscillatory processes of stable systems.
10. How are transient quality requirements represented graphically?

## **PRACTICAL WORK**

### **Study of Typical Links of ACS**

Study questions:

1. Study of the parameters of the typical links of the ACS.

Purpose of the work:

1. Study of the relationship between the parameters of the typical links of ACS and their characteristics.
2. Acquisition of skills in the study of time and frequency characteristics of typical ACS links.

Problem statement

The object of the study is the typical links of the ACS with one input and one output. At the same time, the model of a one-dimensional ACS is given in the form of a complex transfer function recorded as the ratio of polynomials

$$W(s) = \frac{b_m s^m + \dots + b_1 s + b_0}{a_n s^n + \dots + a_1 s + a_0}.$$

It is necessary to obtain and study the main dynamic and frequency characteristics of typical ACS links for your version.

#### **Brief information from the theory**

A typical dynamic link of an ACS is a component of the system, which is described by a differential equation not higher than the second order. A link, as a rule, has one input and one output. According to their dynamic properties, typical links are divided into the following varieties: positional, differentiating and integrating. Positional (static) links are those links in which a linear relationship between input and output signals is observed in a steady state. At a constant level of the input signal, the signal at the output also tends to a constant value.

Differentiating links are those in which the output signal in the steady-state mode is proportional to the time derivative of the input signal.

Integrating links are those whose output signal is proportional to the integral in time from the input signal.

A link is considered given and defined if its transfer function or differential equation is known. In addition, links have time and frequency characteristics.

The transfer function of any ACS in the general case can be represented as a product of the transfer functions of the following form:

$$K, S^n, \frac{1}{T \cdot S + 1}, \frac{1}{T^2 S^2 + 2\xi TS + 1}, \tau \cdot S + 1, \tau^2 S^2 + 2\xi \tau S + 1$$

where K, n, T, ,  $\xi$  are  $\tau$  constants, and  $K > 0$ ,  $n > 0$ ,  $T > 0$ ,  $0 < \xi < 1$ ,  $\tau > 0$ .

These transfer functions define typical dynamic links. The transfer functions and timing characteristics of typical links are shown in Table 1. The frequency characteristics of typical links are shown in Table 2.

In Tables. 1 and 2 only the characteristics of the main typical links are indicated.

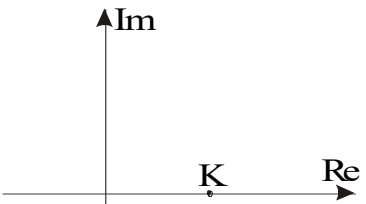
Special dynamic links of ACS. Special dynamic links of ACS are stable non-minimum-phase links (contain zeros in the right half-plane, and poles are negative real numbers), unstable links (contain poles in the right half-plane), links with distributed parameters: irrational and transcendental (the number of zeros and poles tends to infinity)

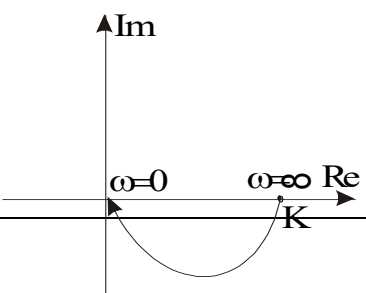
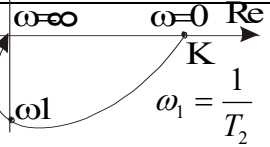
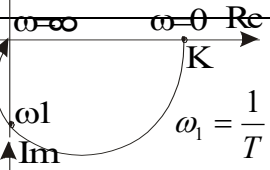
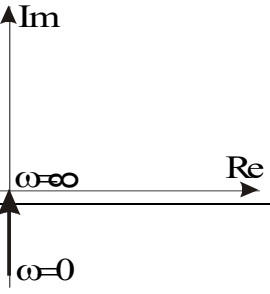
Table 1. – Temporal characteristics of typical links

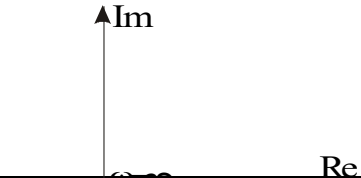
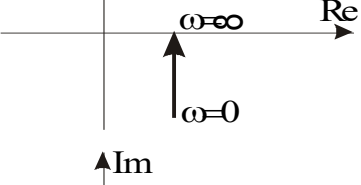
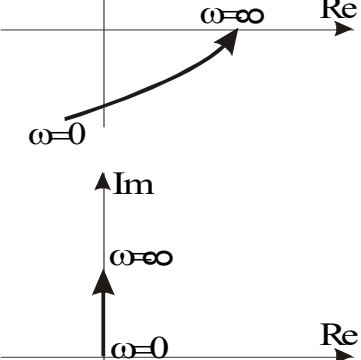
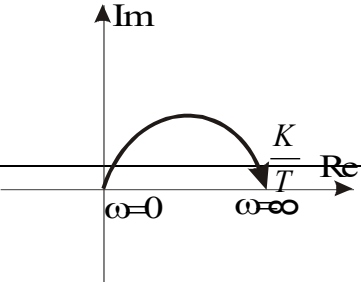
| Link Type                                                   | Transfer functions                                                                                                            | Temporary functions                                                                                                                                                                                                                                                      |
|-------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Positional links                                            |                                                                                                                               |                                                                                                                                                                                                                                                                          |
| 1. Amplifying                                               | $W = K$                                                                                                                       | $h(t) = K \cdot 1(t) \quad \omega(t) = K \cdot \delta(t)$                                                                                                                                                                                                                |
| 2. Aperiodic 1st order                                      | $W = \frac{K}{TS + 1}$                                                                                                        | $h(t) = K(1 - e^{-\frac{t}{T}}) \quad \omega(t) = \frac{K}{T} e^{-\frac{t}{T}}$                                                                                                                                                                                          |
| 3. Aperiodic of the 2nd order (inertial)<br>$T_1 \geq 2T_2$ | $W = \frac{K}{T_2^2 S^2 + T_1 S + 1} = \frac{K}{(T_3 S + 1)(T_4 S + 1)}$<br>$, T_3, T_4 = 0,5(T_1 \pm \sqrt{T_1^2 - 4T_2^2})$ | $h(t) = K \left( 1 - \frac{1}{T_3 - T_4} \left( T_3 e^{-\frac{t}{T_3}} - T_4 e^{-\frac{t}{T_4}} \right) \right)$<br>$\omega(t) = \frac{k}{T_3 - T_4} \left( e^{-\frac{t}{T_3}} - e^{-\frac{t}{T_4}} \right)$                                                             |
| 4. Oscillatory<br>$0 < \xi < 1$                             | $W = \frac{K}{T^2 S^2 + 2\xi TS + 1}$                                                                                         | $h(t) = K \left( 1 - \frac{1}{\lambda T} e^{-\frac{\xi}{T} t} \sin(\lambda t + \theta) \right)$<br>$\lambda = \frac{1}{T} \sqrt{1 - \xi^2} \quad \theta = \text{arctg} \frac{\lambda T}{\xi}$<br>$\omega(t) = \frac{K}{\lambda T^2} e^{-\frac{\xi}{T} t} \sin \lambda t$ |
| 5. Conservative                                             | $W = \frac{K}{T^2 S^2 + 1}$                                                                                                   | $h(t) = K \left( 1 - \cos \frac{t}{T} \right) \quad \omega(t) = \frac{K}{T} \sin \frac{t}{T}$                                                                                                                                                                            |

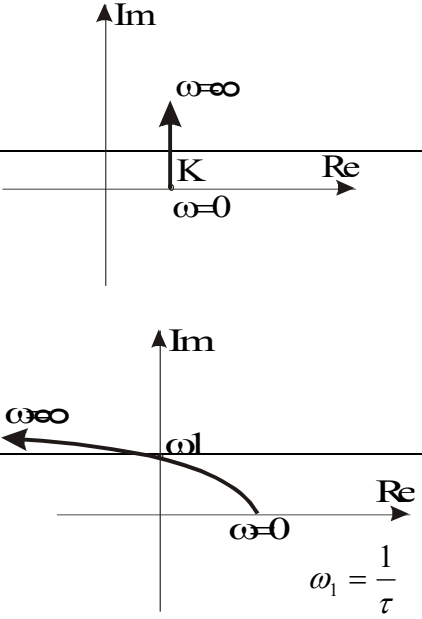
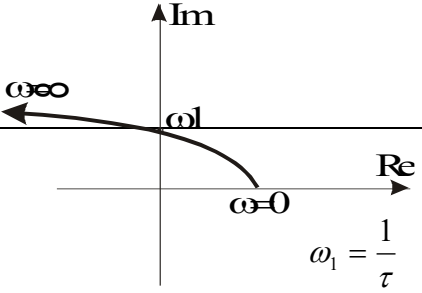
| Integrating links            |                                                                                                                            |                                                                                                                            |
|------------------------------|----------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------|
| 6. Integrating Ideal         | $W = \frac{K}{S}$                                                                                                          | $h(t)=k \cdot t \quad \omega(t)=k \cdot 1(t)$                                                                              |
| 7. Integrating inertial      | $W = \frac{K}{S(TS + 1)}$                                                                                                  | $h(t) = K \left( t - T \left( 1 - e^{-\frac{t}{T}} \right) \right)$<br>$\omega(t) = K \left( 1 - e^{-\frac{t}{T}} \right)$ |
| 8. Isodrome of the 1st order | $W = \frac{K(\tau S + 1)}{S} = K_1 + \frac{K}{S} K_1 = K\tau$                                                              | $h(t) = K_1 + K \cdot t$<br>$\omega(t) = K_1 \delta(t) + K$                                                                |
| 9. Isodrome of the 2nd order | $W = \frac{K(\tau^2 S^2 + 2\xi\tau S + 1)}{S^2} = K_2 + \frac{K_1}{S} + \frac{K}{S^2} K_1 = K2\xi\tau \quad K_2 = K\tau^2$ | $h(t) = K_2 + K_1 t + K t^2$<br>$\omega(t) = K_2 \delta(t) + K_1 + 2Kt$                                                    |
| Differentiating links        |                                                                                                                            |                                                                                                                            |
| 10. Perfect differentiating  | $W=KS$                                                                                                                     | $h(t) = K\delta(t)$<br>$\omega(t) = K\delta'(t)$                                                                           |
| 11. Differentiating inertial | $W = \frac{KS}{TS + 1}$                                                                                                    | $h(t) = \frac{K}{T} e^{-\frac{t}{T}}$<br>$\omega(t) = \frac{K}{T} \delta(t) - \frac{K}{T} e^{-\frac{t}{T}}$                |
| 12. Forcing of the 1st order | $W = K(\tau S + 1)$                                                                                                        | $h(t) = K + \tau K \delta(t)$<br>$\omega(t) = K\delta(t) + K\tau\delta'(t)$                                                |
| 13. Forcing of the 2nd order | $W = k(T^2 S^2 + 2\xi Ts + 1)$                                                                                             |                                                                                                                            |

Table 2. – Frequency characteristics of links

| Frequency Transfer Function | Amplitude M(ω) and phase φ(ω) characteristics | Amplitude-phase frequency response                                                    |
|-----------------------------|-----------------------------------------------|---------------------------------------------------------------------------------------|
| 1. Amplifying<br>(jω)=K     | M(ω)=K    φ(ω)=0                              |  |

|                                                                                                                        |                                                                                                                                                  |                                                                                       |
|------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------|
| <p>2. First-order aperiodic</p> $W(j\omega) = \frac{K}{1 + j\omega T}$                                                 | $M(\omega) = \frac{K}{\sqrt{1 + \omega^2 T^2}}$ $\varphi(\omega) = -\arctg \omega T$                                                             |    |
| <p>3. Aperiodic second-order (inertial)</p> $W(j\omega) = \frac{K}{(1 - \omega^2 T_2^2) + j\omega T_1}$                | $M(\omega) = \frac{K}{\sqrt{1 + \omega^2 (T_1^2 - 2T_2^2) + \omega^4 T_2^4}}$ $\varphi(\omega) = -\arctg \frac{\omega T_1}{1 + \omega^2 T_2^2}$  |    |
| <p>4. Oscillatory<br/><math>0 &lt; \xi &lt; 1</math></p> $W(j\omega) = \frac{K}{(1 - \omega^2 T^2) + j\omega^2 \xi T}$ | $M(\omega) = \frac{K}{\sqrt{1 + \omega^2 2T^2 (2\xi^2 - 1) + \omega^4 T^4}}$ $\varphi(\omega) = -\arctg \frac{\omega^2 \xi T}{1 - \omega^2 T^2}$ |    |
| <p>5. Conservative</p> $W(j\omega) = \frac{K}{1 - \omega^2 T^2}$                                                       | $M(\omega) = \frac{K}{1 - \omega^2 T^2}$ $\varphi(\omega) = 0$                                                                                   |  |
| <p>6. Integrating Ideal</p> $W(j\omega) = \frac{K}{j\omega}$                                                           | $M(\omega) = \frac{K}{\omega}$ $\varphi(\omega) = -90^\circ$                                                                                     |  |

|                                                                                                              |                                                                                                                                                                              |                                                                                       |
|--------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------|
| <p>7. Integrating inertial</p> $W(j\omega) = \frac{K}{-\omega^2 T + j\omega}$                                | $M(\omega) = \frac{K}{\omega \sqrt{1 + \omega^2 T^2}}$ $\varphi(\omega) = -(90^\circ + \arctg \omega T)$                                                                     |    |
| <p>8. First-order isodrome</p> $W(j\omega) = \frac{K(1 + j\omega\tau)}{j\omega}$                             | $M(\omega) = \frac{K}{\omega} \sqrt{1 + \omega^2 \tau^2}$ $\varphi(\omega) = -90^\circ + \arctg \omega \tau$                                                                 |    |
| <p>9. Second-order isodrome</p> $W(j\omega) = \frac{K((1 - \omega^2 \tau^2) + j\omega 2\xi\tau)}{-\omega^2}$ | $M(\omega) = \frac{K}{\omega^2} \sqrt{1 + \omega^2 2\tau^2(2\xi^2 - 1) + \omega^4 \tau^4}$ $\varphi(\omega) = \arctg \frac{2\omega\tau\xi}{1 - \omega^2 \tau^2} - 180^\circ$ |    |
| <p>10. Perfect differentiating</p> $W(j\omega) = jK\omega$                                                   | $M(\omega) = k\omega$ $\varphi(\omega) = 90^\circ$                                                                                                                           |  |
| <p>11. Differentiating inertial</p> $W(j\omega) = \frac{jK\omega}{1 + j\omega T}$                            | $M(\omega) = \frac{k\omega}{\sqrt{1 + \omega^2 T^2}}$ $\varphi(\omega) = 90^\circ - \arctg \omega T$                                                                         |  |

|                                                                                           |                                                                                                                                                  |                                                                                     |
|-------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|
| 12. Forcing of the 1st order<br>$W(j\omega) = K(1 + j\omega\tau)$                         | $M(\omega) = K\sqrt{1 + \omega^2\tau^2}$<br>$\varphi(\omega) = \arctg\omega\tau$                                                                 |  |
| 13. Forcing of the 2nd order<br>$W(j\omega) = K((1 - \omega^2\tau^2) + j\omega 2\xi\tau)$ | $M(\omega) = K\sqrt{1 + \omega^2 2\tau^2 (2\xi^2 - 1) + \omega^4\tau^4}$<br>$\varphi(\omega) = \arctg \frac{2\omega\tau\xi}{1 - \omega^2\tau^2}$ |  |

14. Stable non-minimum-phase links:

$$1. W(p) = \frac{k(1 - pT_1)}{1 + pT_2}; \quad 2. W(p) = \frac{k(1 - pT_1)(1 - pT_2)}{1 + pT_3};$$

15. Unstable links:

$$1. W = \frac{K}{-T_2^2 S^2 + T_1 S + 1}; \quad 2. W = \frac{K}{T_2^2 S^2 - T_1 S + 1}; \quad 3. W = \frac{K}{T_2^2 S^2 + T_1 S - 1};$$

$$4. W = \frac{K}{1 - 2\xi TS + T^2 S^2}; \quad 5. W = \frac{K}{T^2 S^2 - 1}; \quad 6. W = \frac{K}{S(TS - 1)};$$

$$7. W = \frac{K}{T S - 1}; \quad 8. W = \frac{K}{1 - T S};$$

### Variants of tasks

Explore the following links for your variant

| Variant number | Number of the examined link | Variant number | Number of the examined link |
|----------------|-----------------------------|----------------|-----------------------------|
| 1              | 8,3,7,11, 14.1,15.1         | 11             | 2,5,9,11,14.1, 15.3         |
| 2              | 2,4,7,8, 14.2, 15.2         | 12             | 3,4,7,8, 14.2, 15.4         |
| 3              | 3,4, 8,9, 14.1,15.3         | 13             | 4,2,8,6,14.1, 15.5          |
| 4              | 4,5,7,9, 14.2, 15.4         | 14             | 5,7,9,11, 14.2, 15.6        |
| 5              | 2,3,8,7, 14.1, 15.5         | 15             | 1,2,7,11, 14.1, 15.7        |
| 6              | 4,5,7,6, 14.2, 15.6         | 16             | 2,3,8,9,14.2, 15.8          |
| 7              | 3,2,8,11, 14.1, 15.7        | 17             | 3,5,9,7, 14.1, 15.1         |
| 8              | 3,5,9,7, 14.2, 15.8         | 18             | 4,1,7,6, 14.1, 15.2         |
| 9              | 2,4,7,8, 14.1, 15.1         | 19             | 5,2,8,11, 14.2, 15.3        |
| 10             | 3,4,8,9, 14.2, 15.2         | 20             | 1,4,9,5, 14.1, 15.4         |

Note:

1. The numbers of the studied links correspond to the types of links that are given in the section "Brief information from the theory".
2. When examining the amplifier link, put  $k = 1.5; 2.9; 0.3$ . Write down its equation in statics and dynamics. Depict its time and frequency characteristics. Explain whether its characteristics change when the frequency changes from 0 to  $\infty$ .
3. When studying the first-order aperiodic link, analyze the graph of the transient function. Find the relationship between the time constant  $T$  of the aperiodic link and the time it takes for the transient function to approach its steady value by an amount less than 5% of this value. Consider options with different values of the gain factor and time constant (0.01; 0.1; 1, 5.0; 10.0; 100.0 sec) of the aperiodic link. Draw conclusions.
4. Second-order aperiodic link. Investigate the link, taking  $T_1 = 0.01; 0.1; 0.5; 1.0; 5.0$  and taking into account that  $T_1 \geq 2T_2$ . Draw conclusions.
5. Study of the oscillatory link. Determine the dependence of the link transient processes on the parameter  $\xi$  - the attenuation parameter (relative attenuation coefficient or damping index). The attenuation parameter lies within the limits  $0 < \xi < 1$  of.  $\xi = 0.1; 0.3; 0.7; 0.9; 1.0$ .  $\xi$
6. Study of the conservative link. Examine the link according to the method set out in the sequence of work. Make sure that the conservative link is a special case of the second-order static link when  $\xi = 0$ . Determine at what frequency the amplitude characteristic has a break.
7. Integrating ideal link. Examine the link according to the method given below. Set the value of  $k$  independently. Determine the link LAH (its inclination), as well as the dependence of the PHC on frequency. Determine whether the PHC changes its position depending on the change in the coefficient  $k$ . Determine whether the integrating link is lagging behind or ahead in the phase of the signal, and in what frequency range.
8. Integrating inertial link. Examine the characteristics of the link, setting the values  $k$  and  $T$  independently. Visualize the graphically integrating link as a sequential connection of two links: an integrating link and a static link of the first order. Get the

characteristics of these links and compare them with the obtained characteristics of the integrating inertial link. Draw conclusions.

9. 1st Order Isodrome Link Examine the characteristics of the link by setting the  $k$  and  $T$  values yourself. Graphically represent the 1st order isodrome link as a parallel connection of the ideal integrating link and the amplifying link. Get the characteristics of these links and compare them with the obtained characteristics of the 1st order isodrome link. Draw conclusions.

10. Isodrome link of the 2nd order. Examine the characteristics of the link, setting the values of  $k$  and  $T$  independently. Graphically represent the isodrome link of the 2nd order as a parallel connection of three typical links: U-link ( $k_2$ ), I-link ( $k_1/s$ ), link ( $k/s^2$ ).

11. Perfect differentiating link. Get the characteristics of the link and compare them with the characteristics of the integrating link.

12. Differentiating inertial link. Examine the characteristics of the link. Imagine this link as a sequential connection of an ideal differentiating link and a static link of the first order. Examine these links. Show that the resulting characteristic of this link is obtained from the addition of the characteristics of typical links.

13. First-order forcing link. Examine the characteristics of the link and compare them with the first-order aperiodic link. Draw conclusions.

14. Second-order forcing link. Examine the characteristics of the link when  $\xi < 1$ ,  $\xi = 0$ ,  $\xi > 1$ . comparing the obtained characteristics with the characteristics of the oscillating link, the characteristics of aperiodic links.

### **Sequence of work:**

1. For your version of the transfer function of the link, write down its differential equation.

2. Using the syntax of the previous practical work, create an LTI object.

3. Find zeros and poles (singular points) of the transfer function and their location on the complex plane. Draw a conclusion about the stability of the studied link.

4. Calculate the link transfer coefficient.

5. Determine the complex value of the transmission coefficient for frequencies of 0.1; 1; 10; 100 rad/s.
6. Construct the transient and impulse transient characteristics of the studied link. Analyze the obtained characteristics.
7. Construct logarithmic amplitude and frequency responses, Nyquist hodograph. Examine the obtained characteristics and determine the features of each link (for your own variant). Determine whether the AFC covers the point  $(-1, j0)$ . Mark trajectories for positive and negative frequencies, points at which the frequency tends to zero and to infinity.
8. Determine the margin of stability of the link in phase and amplitude.
9. Calculate the maximum amplitude value and the corresponding frequency value.
10. To study the influence of parameters on time and frequency characteristics, on the position of special points of transfer functions. The results should be presented in graphic form, conclusions should be formulated and recorded.
11. Answer control questions.
12. Draw up a report.
13. Submit the report to the teacher and defend the work.

The report is drawn up in accordance with the requirements for the design of laboratory work at the university, and must contain a title page, a statement of the purpose of the work, a statement of the task in accordance with the version of the task, the results of the work, and conclusions.

### **Control questions:**

1. What are the characteristics of the classification of SPG links?
2. What are the simplest links that can be singled out when considering the established modes of operation?
3. Name typical links that differ in dynamic properties.
4. What is the main difference between stable and unstable links?
5. Explain the difference between minimal-phase and non-minimum-phase links?

6. Write down the differential equations and transfer functions of typical links?
7. Draw the transient and frequency characteristics of typical links and compare them for stable and unstable, as well as for minimum phase and non-minimum phase links.

## PRACTICAL WORK

### Investigation of Stability of Linear Systems Using Algebraic Methods

#### Purpose of the work:

1. To study algebraic methods for studying the stability of linear systems.
2. To acquire practical skills in solving problems of system synthesis using Matlab PPP.

#### Study Questions

1. Determination of the stability of linear systems by the location of the roots of the characteristic equation.
2. Determination of stability of linear one-dimensional and multidimensional systems using the Raus-Hurwitz criterion.

#### Brief information from the theory

##### The concept of sustainability

**Stability** is the property of a system to return to the initial or close to it established regime after any exit from it as a result of any impact.

Fig. 1 shows typical curves of transient processes in unstable (Fig. 1, a) and stable (Fig. 1, b) systems. If the system is unstable, then any shock is enough to start a divergent process of departure from the initial steady state. This process can be aperiodic (curve 1 in Fig. 1, a) or oscillatory (curve 2 in Fig. 1, a).

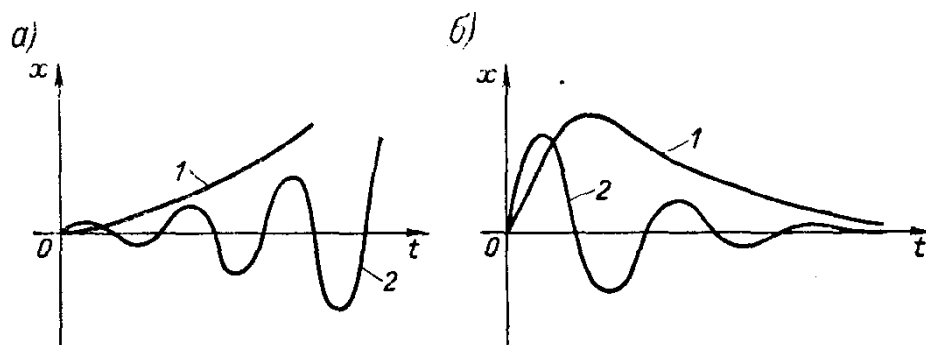


Figure 1. - On the concept of system stability

An aperiodic divergent process can, for example, occur in an ACS if the polarity of the impact on the object is mistakenly switched in its control device, as a result of which the control device will provide not negative, but positive feedback around the object. In this case, the control device will not eliminate the deviation  $X$ , but will act in the opposite direction, causing its avalanche-like change.

An oscillatory divergent process can occur with an unlimited increase in the transmission coefficient of the system, as a result of which the control device will excessively energetically affect the object, trying to eliminate the initial deviations  $X$ . In this case, with each successive return of  $X$  to zero under the action of the control device, the curve  $X$  will cross the  $x$ -axis with increasing speed and the process as a whole will diverge.

In the case of a stable system (Fig. 1, b), the transient process caused by any impact eventually attenuates aperiodically (curve 1) or oscillatively (curve 2), and the system returns to a steady state.

**Thus, a stable system can be defined as a system in which transient processes are dampened.**

However, a system can operate under conditions of continuously changing influences, when there is no steady regime at all. With this in mind, we will give the following, more general definition of stability: **a system is stable if its output value remains limited under the conditions of action of perturbations of limited magnitude on the system.**

Let us consider what the stability of the system depends on, what determines it. For this purpose, let us turn to the equation of system dynamics

$$x = W_s(p)f ,$$

where

$$W_s(p) = \frac{W_{xj}(p)}{1 + W(p)}; \quad W(p) = \frac{R(p)}{Q(p)}; \quad W_{xj} = \frac{R_{xj}(p)}{Q_{xj}(p)}.$$

Having freed ourselves from fractions in the numerator and denominator of the

transfer function, we can represent it as follows:  $W_s(p) = M(p)/D(p)$  and, accordingly, switch to the usual form of notation in the form of a differential equation

$$D(p)x = M(p)f \quad (1)$$

The solution of this linear inhomogeneous equation in general consists of two components:  $x(t) = x_{ust}(t) + xp(t)$ . (2)

Here  $x_{ust}(t)$  is a partial solution of the inhomogeneous equation (1) with the right side, describing the forced regime of the system established at the end of the transient process;

$xp(t)$  is the general solution of the homogeneous equation  $D(p)x=0$ , describing the transient process in the system caused by a given perturbation.

**The system will be stable if the transients  $xp(t)$  caused by perturbations are dampened, that is, if  $xp(t)$  tends to zero over time.**

The solution  $xp(t)$  of a homogeneous differential equation has the form:

$$x_n(t) = \sum_{i=1}^n C_i e^{\lambda_i t}. \quad (3)$$

Here  $C_i$  are constant integrations determined by initial conditions and perturbation,  $\lambda_i$  are the roots of the characteristic equation

$$D(\lambda) = 0, \quad (4)$$

where the polynomial  $D(\lambda)$ , called the characteristic polynomial, is the left side of equation (1) of the dynamics of the system after the differentiation operator  $p$  has been replaced by a complex variable  $\lambda$ .

The polynomial  $D(\lambda)$  is the denominator of the transfer function  $W3(p)$  of the system after it has been freed from the fraction and replaced by  $\lambda$ , i.e.

$$D(\lambda) = R(\lambda) + Q(\lambda), \quad (5)$$

where  $R(\lambda)$  and  $Q(\lambda)$  are the numerator and denominator of the transfer function  $W(p)$  of the open system when **replacing  $p$  with  $\lambda$** .

Thus, the transient process  $xp(t)$  is the sum of the components, the number of which is determined by the number of roots  $\lambda$  of the characteristic equation (4), that is, by the order of the system's equation.

In general, roots  $\lambda$  are complex. At the same time, they form pairs of conjugate roots:

$$\lambda_{i,i+1} = \alpha_i \pm j\beta_i,$$

where  $\alpha_i$  can be a positive or negative value. Each such pair of roots gives in expression (3) a transient component equal to

$$C_i e^{(\alpha_i + j\beta_i)t} + C_{i+1} e^{(\alpha_i - j\beta_i)t} = C'_i e^{\alpha_i t} \sin(\beta_i t + \varphi_i),$$

where  $C'_i$  and  $\varphi_i$  are defined by  $C_i$  and  $C_{i+1}$ .

This component is a sinusoid with an amplitude that varies exponentially in time. At the same time, if  $\alpha_i < 0$ , this component will be attenuated. On the contrary, if  $\alpha_i > 0$ , divergent oscillations will be obtained. If  $\alpha_i = 0$ , which corresponds to a pair of imaginary roots, there will be continuous sinusoidal oscillations.

The condition for the attenuation of this component of the transition process is the negativity of the real part  $\alpha_i$  of the corresponding pair of conjugate roots of the characteristic equation.

In the special case, when  $\beta_i = 0$ , we have the real root  $\lambda = \alpha_i$ . The corresponding component of the transition  $C_i e^{\lambda_i t}$  is an exponent that will decay or increase depending on the sign  $\alpha_i$ .

Thus, in the general case, the transient process in a system consists of oscillatory and aperiodic components. Each oscillatory component owes its appearance to a pair of complex conjugation of roots, and each aperiodic component to the real root. It follows that the **general condition for the attenuation of all components, and hence of the entire transient process as a whole, is the negativity of the real parts of all the roots of the characteristic equation of the system, that is, all the poles (roots of the denominator) of the transfer function of the system.**

If at least one root has a positive real part, it will give a divergent component of the transient process, and the system will be unstable. The presence of a pair of conjugate purely imaginary roots  $\lambda_{i,i+1} = \pm j\beta_i$  will give an unquenchable harmonic component of the transient process. At the same time, unquenchable oscillations with a frequency equal to  $\beta_i$  will be established in the system.

If we depict the roots of the characteristic equation of a system by points on a complex plane (Fig. 2), then the general condition for the stability of a linear system found above can also be formulated as follows: the condition for the stability of the system is the location of all the roots of the characteristic equation, i.e., the poles of the transfer function of the system, in the left complex half-plane, or all of them must be left.

The presence of a root on the imaginary axis means that the system is on the border of stability.

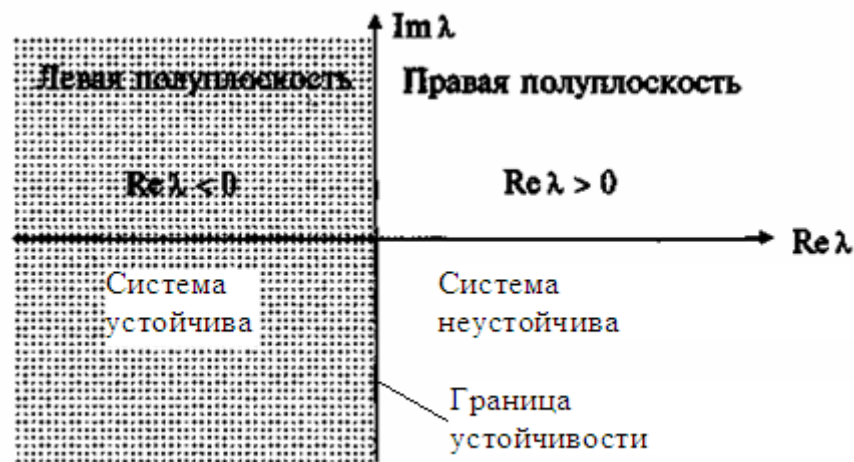


Figure 2. – Location of the roots of the ACS characteristic equation

### Raus-Hurwitz stability criterion

This is an algebraic criterion according to which the stability conditions are reduced to the fulfillment of a series of inequalities that connect the coefficients of the system's equation. In various forms, this criterion was proposed by the English mathematician Rous and then by the Swiss mathematician Hurwitz at the end of the last century. Let us cite this criterion without proof in the form of Hurwitz.

Take the characteristic polynomial that defines the left-hand side of equation (1) of the system,  $D(\lambda) = \alpha_0 \lambda^n + \alpha_1 \lambda^{n-1} + \dots + \alpha_{n-1} \lambda + \alpha_n$  (6)

where we assume  $\alpha_0 > 0$ , which can always be achieved by multiplying the polynomial by  $-1$ , if necessary. Let us make a determinant from the coefficients of this polynomial.

$$\Delta_n = \begin{vmatrix} \alpha_1 & \alpha_3 & \alpha_5 & \dots & \dots & \dots & 0 \\ \alpha_0 & \alpha_2 & \alpha_4 & \alpha_6 & \dots & \dots & 0 \\ 0 & \alpha_1 & \alpha_3 & \alpha_5 & \dots & \dots & 0 \\ 0 & \alpha_0 & \alpha_2 & \alpha_4 & \alpha_6 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \alpha_{n-1} & 0 \\ 0 & \dots & \dots & \dots & \dots & \alpha_{n-2} & \alpha_n \end{vmatrix} \quad (7)$$

This determinant is called the Hurwitz determinant. It has **n** rows and **n** columns. The first row contains all odd coefficients to the last, after which the row is filled to the prescribed number of **n** elements with bullets. The second row includes all even coefficients and also ends with zeros. The third row is obtained from the first, and the fourth from the second by shifting to the right by one element. Zero is placed on the vacant space on the left.

As a result, all coefficients except 0 are sequentially in the main diagonal of the determinant.  **$\alpha$**

The condition of stability consists in the requirement of the positivity of the Hurwitz determinant and all its diagonal minors. Let us expand the Hurwitz criterion for several specific values of **n**.

$$\text{For } \mathbf{n} = 1 \ D(\lambda) = \alpha_0 \lambda + \alpha_1$$

and the conditions of stability are reduced to inequalities:  **$\alpha_0 > 0$ ;  $\alpha_1 > 0$** .

Hence, for example, a first-order link with a transfer function  **$k/(Tp+1)$**  is stable, and a link with a transfer function  **$k/(Tp-1)$**  is unstable.

$$\text{For } \mathbf{n} = 2 \ D(\lambda) = \alpha_0 \lambda^2 + \alpha_1 \lambda + \alpha_2 \ \Delta_2 = \begin{vmatrix} \alpha_1 & 0 \\ \alpha_0 & \alpha_2 \end{vmatrix} = \alpha_1 \alpha_2.$$

Stability conditions:  **$\alpha_0 > 0$ ;  $\alpha_1 > 0$ ;  $\alpha_2 > 0$**  (the last inequality is reduced to  $\Delta_{\text{inequality of } 2 > 0}$ , if we take into account the previous inequality  $\alpha_{of 1 > 0}$ ).

For example, a link with a transfer function  **$k/(T^2p^2 + T^2p + 1)$**  is stable if all terms in the denominator are preceded by a plus sign.

$$\text{For } \mathbf{n} = 3 \ D(\lambda) = \alpha_0 \lambda^3 + \alpha_1 \lambda^2 + \alpha_2 \lambda + \alpha_3$$

$$\Delta_3 = \begin{vmatrix} \alpha_1 & \alpha_3 & 0 \\ \alpha_0 & \alpha_2 & 0 \\ 0 & \alpha_1 & \alpha_3 \end{vmatrix}$$

Stability conditions:  $\alpha_0 > 0$ ;  $\alpha_1 > 0$ ;  $\Delta_2 = \alpha_1 \alpha_2 - \alpha_0 \alpha_3 > 0$   $\Delta_3 = \alpha_3 \Delta_2 > 0$ .

The last inequality, taking into account the penultimate condition  $\Delta_2 > 0$ , comes down to the requirement  $\alpha_3 \Delta_2 > 0$ . Thus, in general, these stability conditions consist in the positivity of all odds and the penultimate minor of  $\Delta_2$ . (The need for positivity  $\alpha_2$  follows from the condition  $\Delta_2 > 0$  and the positivity of all other odds).

For  $n = 4$   $D(\lambda) = \alpha_0 \lambda^4 + \alpha_1 \lambda^3 + \alpha_2 \lambda^2 + \alpha_3 \lambda + \alpha_4$

$$\Delta_4 = \begin{vmatrix} \alpha_1 & \alpha_3 & 0 & 0 \\ \alpha_0 & \alpha_2 & \alpha_4 & 0 \\ 0 & \alpha_1 & \alpha_3 & 0 \\ 0 & \alpha_0 & \alpha_2 & \alpha_4 \end{vmatrix}.$$

Stability conditions:  $\alpha_0 > 0$ ;  $\alpha_1 > 0$ ;  $\Delta_2 = \alpha_1 \alpha_2 - \alpha_0 \alpha_3 > 0$   $\Delta_3 = \alpha_3 \Delta_2 - \alpha_{12} \alpha_4 > 0$

$$\Delta_4 = \alpha_4 \Delta_3 > 0.$$

Obviously, the stability conditions boil down to the requirement of the positivity of all coefficients and the penultimate minor of  $\Delta_3$ . (The condition  $\Delta_2 > 0$  follows from the inequality  $\Delta_{of3} > 0$ , given that  $\alpha_4 > 0$ .)

For  $n = 5$   $D(\lambda) = \alpha_0 \lambda^5 + \alpha_1 \lambda^4 + \alpha_2 \lambda^3 + \alpha_3 \lambda^2 + \alpha_4 \lambda + \alpha_5$

The conditions of stability, if we act in the same way, will be reduced here to the positive of all coefficients and two minors:  $\Delta_2$  and the penultimate  $\Delta_4$ .

It has been proved that in the general case for a system **of order n**, for a condition of stability, there is a requirement for the positivity of all coefficients of the equation. The analysis of stability should begin with the verification of this simple **necessary but insufficient condition of stability**.

The stability conditions obtained from the Raus-Hurwitz criterion become more complex with the growth of the system's order. At the same time, for systems of sufficiently high order, it is difficult to find out the influence on the stability of the system of the values of individual parameters of the links included in the coefficients of the equation. This is due to the fact that, as a rule, the same parameters are

simultaneously included in several coefficients of the system's equation. Therefore, the Raus-Hurwitz criterion is used only for systems of low order, and, first of all, for stability analysis, when it is necessary to determine whether the system is stable at known values of all its parameters. When solving the problem of system synthesis, when it is necessary to select the values of individual parameters of the system, the Rous-Hurwitz criterion becomes inconvenient already for systems above the fourth order.

As an example, let us consider a third-order system consisting of three static links of the first order. The transfer function of this system is as follows:

$$W(p) = \frac{k_1 k_2 k_3}{(T_1 p + 1)(T_2 p + 1)(T_3 p + 1)} \quad (8)$$

In this case,

$$D(\lambda) = R(\lambda) + Q(\lambda) = k_1 k_2 k_3 + (T_1 \lambda + 1)(T_2 \lambda + 1)(T_3 \lambda + 1)$$

i.e. 
$$D(\lambda) = \alpha_0 \lambda^3 + \alpha_1 \lambda^2 + \alpha_2 \lambda + \alpha_3$$

Здесь  $\alpha_0 = T_1 T_2 T_3$ ;  $\alpha_1 = T_1 T_2 + T_2 T_3 + T_3 T_1$ ;

$$\alpha_2 = T_1 + T_2 + T_3; \quad \alpha_3 = k_1 k_2 k_3 + 1 = k + 1;$$

where

$$k = k_1 k_2 k_3$$

The stability condition, as shown above, for  $n = 3$  is reduced to the following inequalities:

$$(8a) \quad \left. \begin{aligned} T_1 T_2 T_3 > 0; \quad T_1 T_2 + T_2 T_3 + T_3 T_1 > 0; \quad T_1 + T_2 + T_3 > 0; \\ k + 1 > 0; \\ \alpha_1 \alpha_2 - \alpha_0 \alpha_3 = (T_1 T_2 + T_2 T_3 + T_3 T_1)(T_1 + T_2 + T_3 - T_1 T_2 T_3 (1 + K)) > 0. \end{aligned} \right\}$$

The first three inequalities are of no interest if we limit the consideration to positive values of time constants.

The next, fourth, inequality imposes a restriction on the negative value of the transfer coefficient  $\mathbf{k}$ . The absolute value of  $\mathbf{k}$  must be less than one. In practice, this inequality is also irrelevant, showing only that the system loses stability when the feedback is incorrectly closed, when it is positive and not negative, that is, when  $\mathbf{k} < 1$ .

The real constraints on the values of the system parameters are imposed by the last inequality (8a). It is convenient to rewrite it as follows:

$$k < (T_1 + T_2 + T_3) (1/T_1 + 1/T_2 + 1/T_3) - 1. \quad (8b)$$

This inequality shows that the stability of the system will be disrupted by an unlimited increase in the transfer coefficient **k** at any values of the time constants. The limiting value **of k** for stability is determined by the time constants of the system. According to (8b), this critical value

$$k_{kr} = (T_1 + T_2 + T_3) (1/T_1 + 1/T_2 + 1/T_3) - 1. \quad (8c)$$

### Methodical example

1. The transfer function of the ACS is set

$$W = \frac{s + 2}{3s^3 + 4s^2 + 5s + 3}.$$

1. Create an LTI object named w, to do this, we will run:

```
>> w=tf([1 2],[3 4 5 3])
```

```
Transfer function:
```

```
      s + 2
```

```
-----  
3 s^3 + 4 s^2 + 5 s + 3
```

2. Define the poles and zeros of the transfer function using the pole, zero commands.

```
>> pole(w)
```

```
ans =
```

```
 -0.2639 + 1.0825i  
 -0.2639 - 1.0825i  
 -0.8055
```

```
>> zero(w)
```

```
ans =
```

```
 -2
```

3. On the complex plane, determine the location of the poles and zeros for a given transfer function using the pzmap (w) command in Fig. 4.

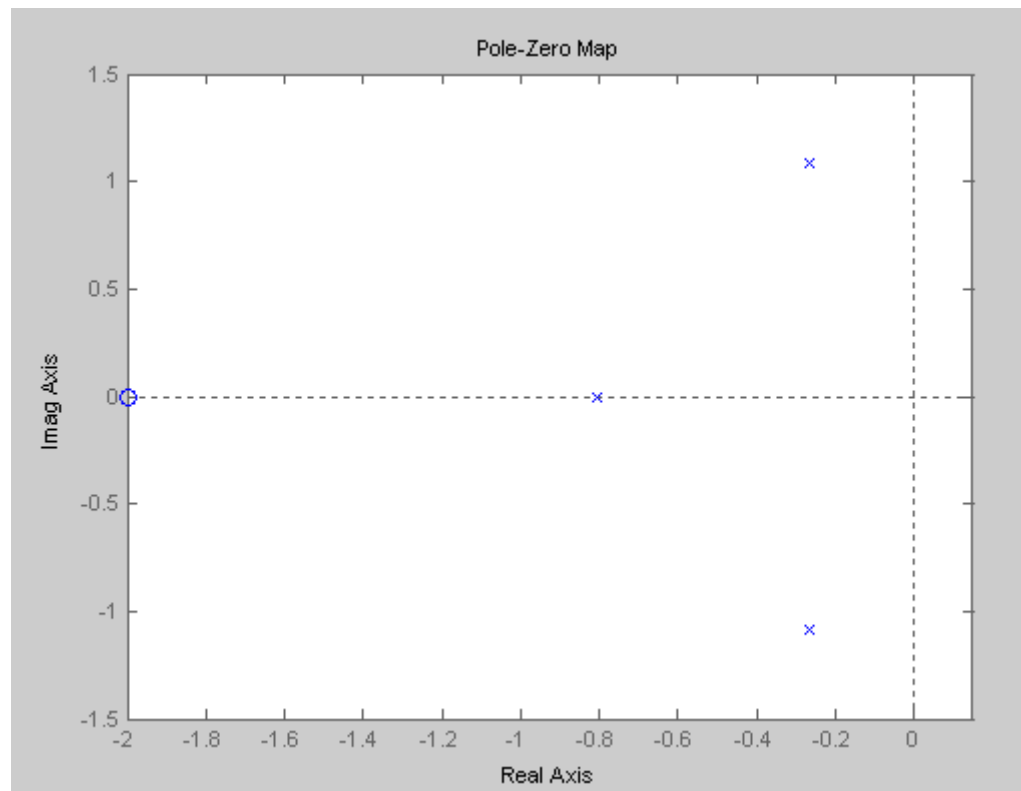


Figure 4. – Location of Poles and Units for a given Transfer Function

In view of the fact that the roots of the denominator (pole) are located in the left half-plane of the complex plane, we conclude that the system under study is stable.

2. The transfer function of the ACS is set

$$W = \frac{s - 3}{2s^2 + s - 3}$$

Let's determine the stability of the system without the help of a computer.

To do this, let's define the poles and zeros of the transfer function:

Poles:

$$2s^2 + s - 3 = 0$$

$$\mathcal{D} = b^2 - 4ac$$

$$\mathcal{D} = 1 - 4 \cdot 2 \cdot (-3) = 25$$

$$s_{1,2} = \frac{-b \pm \sqrt{\mathcal{D}}}{2a}$$

$$s_1 = \frac{-1-5}{4} = -1,5.$$

$$s_2 = \frac{-1+5}{4} = 1;$$

Zeros:

$$s - 3 = 0$$

$$s = 3$$

In order for the system to be stable, it is necessary that all the roots of the denominator (pole) are negative. In this case, we have 1 positive and 1 negative pole. Therefore, our system is unstable.

Now let's determine the stability of the system using MatLab.

1. Create an LTI object named w, to do this, we will run:

```
>> w=tf([0 1 -3],[2 1 -3])
```

```
Transfer function:
```

```
  s - 3
```

```
-----
2 s^2 + s - 3
```

2. Define the poles and zeros of the transfer function using the pole, zero commands.

```
>> pole(w)
```

```
ans =
```

```
 -1.5000
```

```
  1.0000
```

```
>> zero(w)
```

```
ans =
```

```
  3
```

3. On the complex plane, determine the location of the poles and zeros for a given transfer function using the pzmap (w) command in Fig. 5.

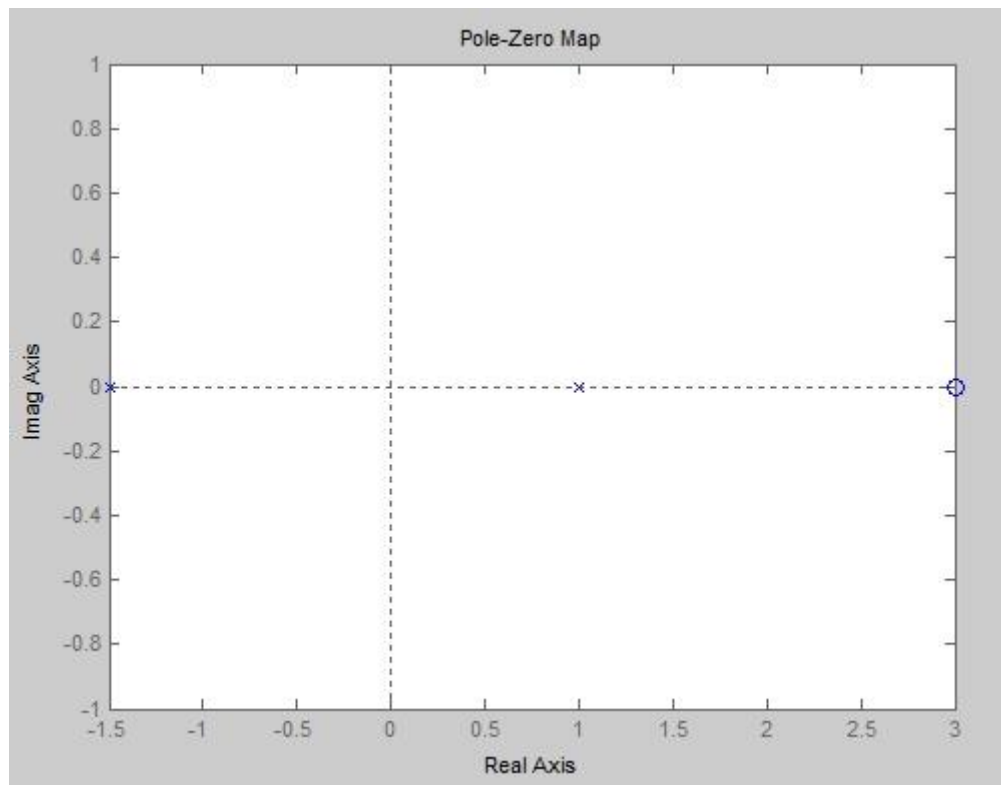


Figure 5. – Arrangement of Poles and Units for a given Transfer Function

In view of the fact that 1 of the 2 roots of the denominator (pole) is located in the right half-plane of the complex plane, we conclude that the system under study is unstable.

4. Let us determine the stability of the system using the Raus-Hurwitz criterion.

$$W = \frac{4s^2 + 2s + 1}{3s^3 + 2s^2 + 9s + 1}$$

According to the transfer function of the system, we make a characteristic polynomial  $D(\lambda) = \alpha_0 \lambda^n + \alpha_1 \lambda^{n-1} + \dots + \alpha_{n-1} \lambda + \alpha_n$

For our system,  $\alpha_0 = 3$ ,  $\alpha_1 = 2$ ,  $\alpha_2 = 9$ ,  $\alpha_3 = 1$ .

$$D(\lambda) = 3\lambda^3 + 2\lambda^2 + 9\lambda + 1$$

The requirement that all coefficients of the equation are positive is met. Now, in order to establish the stability of the system, it is necessary to check the requirement of the positive Hurwitz determinant and all its diagonal minors.

$$\Delta_3 = \begin{vmatrix} \alpha_1 & \alpha_3 & 0 \\ \alpha_0 & \alpha_2 & 0 \\ 0 & \alpha_1 & \alpha_3 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} \alpha_1 & \alpha_3 \\ \alpha_0 & \alpha_2 \end{vmatrix}, \quad \Delta_1 = \alpha_1$$

Minor systems can be defined using MatLab.

```
>> D3=[2 1 0;3 9 0;0 2 1]
```

```
D3 =
```

```

     2     1     0
     3     9     0
     0     2     1

```

```
>> det(D3)
```

```
ans =
```

```
15
```

```
>> D2=[2 1;3 9]
```

```
D2 =
```

```

     2     1
     3     9

```

```
>> det(D2)
```

```
ans =
```

```
15
```

That is  $\Delta_3 = 15 > 0$ , ,  $\Delta_2 = 15 > 0$   $\Delta_1 = 2 > 0$ , .

Thus, the requirement of the positivity of the Hurwitz determinant and all its diagonal minors was also fulfilled. Consequently, the system is stable.

### **Procedure for performing practical work**

For your option, determine the stability of the system using algebraic methods:

1. Determine the stability of the system by the location of the roots of the ACS characteristic equation:

- Create an LTI object named w.
- Determine the poles and zeros of the transfer function using the pole, zero commands;
- On the complex plane, determine the location of the poles and zeros using the pzmap(w) command;

- Draw a conclusion about the stability of the system under study.
- 2. Determine the stability of the system using the Raus-Hurwitz criterion:
  - according to the transfer function of the system, imagine the characteristic polynomial  $D(\lambda)$ ;
  - check the requirement for the positive of all coefficients of the equation (a prerequisite);
  - it is necessary to check the requirement of the validity of the Hurwitz determinant and all its diagonal minors;
  - to draw a conclusion about the stability of the system.

#### Variants of tasks

| No Option | Self-propelled guns of the 3rd order           | SPGs of the 4th order                                       |
|-----------|------------------------------------------------|-------------------------------------------------------------|
| 1         | $W = \frac{s^2 + 2s + 1}{4s^3 + s^2 + 3s + 2}$ | $W = \frac{s^3 + 2s + 1}{2s^4 + 3s^3 + 6s^2 + s + 1}$       |
| 2         | $W = \frac{3s^2 + s}{s^3 + 2s^2 + 7s + 1}$     | $W = \frac{s^3 + 5s^2 + 3s + 8}{7s^4 + 3s^3 + 2s^2 + s}$    |
| 3         | $W = \frac{4s^2 + 1}{3s^3 + 2s^2 + s + 4}$     | $W = \frac{5s^2 + s + 6}{2s^4 + 3s^3 + 4s^2 + s + 2}$       |
| 4         | $W = \frac{s + 1}{5s^3 + s^2 + 4s + 1}$        | $W = \frac{s^3 + s + 2}{s^4 + 4s^3 + 2s^2 + 1}$             |
| 5         | $W = \frac{2s + 3}{s^3 + 6s^2 + 1}$            | $W = \frac{s^3 + 4s + 5}{3s^4 + 2s^3 + s^2 + 6s + 1}$       |
| 6         | $W = \frac{s^2 + 4s + 1}{3s^3 + 2s + 4}$       | $W = \frac{2s^3 + 7s^2 + 2s}{s^4 + 5s^3 + 3s^2 + s + 4}$    |
| 7         | $W = \frac{2s^2 + 1}{s^3 + 2s^2 + 7s + 1}$     | $W = \frac{s^2 + 2s}{4s^4 + 5s^3 + 2s^2 + 3s + 1}$          |
| 8         | $W = \frac{3s + 2}{6s^3 + s^2 + 1}$            | $W = \frac{3s^3 + 2}{6s^4 + 2s^3 + 3s^2 + s}$               |
| 9         | $W = \frac{s^2 + 2s + 1}{s^3 + 4s^2 + 2s + 1}$ | $W = \frac{5s^3 + 2s^2 + 1}{6s^4 + 4s^3 + 2s + 1}$          |
| 10        | $W = \frac{2s^2 + 1}{s^3 + 5s + 4}$            | $W = \frac{2s^3 + 3s^2 + 1}{s^4 + 3s^3 + 2s^2 + 5s + 4}$    |
| 11        | $W = \frac{4s^2 + 2}{2s^3 + 3s^2 + 4s}$        | $W = \frac{s^3 + 4s + 2}{2s^4 + 7s^3 + 2s^2 + s + 5}$       |
| 12        | $W = \frac{2s}{3s^3 + s^2 + 8s + 4}$           | $W = \frac{4s^3 + 2s^2 + s + 4}{2s^4 + 4s^2 + 5s + 1}$      |
| 13        | $W = \frac{s^2 + s + 3}{s^3 + 5s^2 + s + 1}$   | $W = \frac{3s^3 + s^2 + 4s + 1}{s^4 + 4s^3 + 5s^2 + s + 7}$ |

|    |                                                  |                                                         |
|----|--------------------------------------------------|---------------------------------------------------------|
| 14 | $W = \frac{3s^2 + 1}{2s^3 + s^2 + 3s + 1}$       | $W = \frac{2s^2 + s + 4}{2s^4 + 2s^3 + 5s^2 + 5s + 1}$  |
| 15 | $W = \frac{s^2}{3s^3 + 2s^2 + 5}$                | $W = \frac{5s^3 + 4s^2}{2s^4 + 6s^3 + s^2 + 5s + 1}$    |
| 16 | $W = \frac{s^2 + 4s + 1}{2s^3 + 2s^2 + 9s + 1}$  | $W = \frac{2s^3 + 3s + 1}{s^4 + 4s^3 + 3s^2 + 2s}$      |
| 17 | $W = \frac{2s + 1}{3s^3 + 5s^2 + 1}$             | $W = \frac{3s^3 + s^2 + 4s + 1}{s^4 + 2s^3 + 3s + 1}$   |
| 18 | $W = \frac{s^2 + 2s}{4s^3 + s^2 + 2s + 3}$       | $W = \frac{3s^3 + s^2}{4s^4 + 2s^3 + 2s^2 + 8s + 1}$    |
| 19 | $W = \frac{4s^2 + 3}{3s^3 + 9s + 1}$             | $W = \frac{s^3 + 6s^2 + 2s + 1}{2s^4 + 3s^2 + s + 3}$   |
| 20 | $W = \frac{6s + 1}{s^3 + 2s^2 + s + 1}$          | $W = \frac{2s^3 + s^2 + 4}{2s^4 + 3s^3 + s^2 + 4s + 1}$ |
| 21 | $W = \frac{2s^2 + 2s + 2}{3s^3 + 4s^2 + 2s + 3}$ | $W = \frac{4s^3 + 3s^2 + 2}{s^4 + 2s^2 + 5s + 3}$       |
| 22 | $W = \frac{4s^2 + s + 1}{3s^3 + 2s^2 + 2}$       | $W = \frac{s^3 + 3s^2 + 4s + 1}{2s^4 + s^3 + 5s + 2}$   |
| 23 | $W = \frac{2s + 1}{4s^3 + s^2 + 3s + 1}$         | $W = \frac{2s^2 + 4s + 1}{2s^4 + s^3 + 3s^2 + s + 1}$   |
| 24 | $W = \frac{s^2 + 2s + 1}{2s^3 + 9s + 2}$         | $W = \frac{5s^2 + 4s + 1}{2s^4 + s^3 + 3s^2 + 2s}$      |
| 25 | $W = \frac{2s^2 + s}{s^3 + 2s^2 + 5s}$           | $W = \frac{s^3 + 8s^2 + 2s}{s^4 + 4s^3 + 3s + 1}$       |
| 26 | $W = \frac{4s^2 + 1}{4s^3 + 2s^2 + s + 1}$       | $W = \frac{3s^3 + 4s^2 + 2s}{5s^4 + 3s + 1}$            |
| 27 | $W = \frac{7s + 3}{s^3 + 3s^2 + 2s + 3}$         | $W = \frac{5s^3 + s^2 + 4s + 9}{2s^4 + 6s^2 + 7s + 1}$  |
| 28 | $W = \frac{4s}{2s^3 + 5s^2 + 3s + 1}$            | $W = \frac{4s^3 + 2s}{s^4 + 2s^3 + 3s + 5}$             |
| 29 | $W = \frac{4s^2 + s + 1}{2s^3 + s^2 + 6s}$       | $W = \frac{2s^2 + 5s + 1}{3s^4 + 3s^3 + s^2 + 6s}$      |
| 30 | $W = \frac{5s + 1}{s^3 + 3s^2 + 2s + 3}$         | $W = \frac{3s^3 + 6}{s^4 + 2s^3 + 3s^2 + 5s + 1}$       |

## PRACTICAL WORK

### Investigation of controllability and observability of linear systems

#### Purpose of the work:

Study of the criteria for the controllability and observability of ACS.

#### Study questions:

1. Determination of the controllability of systems by condition.
2. Determination of the controllability of systems by output.
3. Determination of the observability of systems.

#### Problem statement

A linear multidimensional stationary control system is given, the behavior of which is described by the equations of state and output:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), & x(t_0) &= x_0, \\ y(t) &= Cx(t), & (1)\end{aligned}$$

where  $x$  is the  $n$ -dimensional state vector;  $u$  is the  $r$ -dimensional control vector,  $t \in R^r$ ;  $t \in [t_0, t_1]$  is the time interval of the system's functioning;  $y$  is the  $k$ -dimensional output vector;  $t \in [t_0, t_1]$   $A, B, C$  are matrices of size  $(n \times n)$ ,  $(n \times r)$ ,  $(k \times n)$ , respectively;  $x_0$  is the initial state.

#### Brief information from the theory

A system (1) is said to be fully **controllable in terms of state** if the choice of the control action  $u(t)$  for the interval of time  $[t_0, t_1]$  can transform the system from any initial state  $x(t_0)$  to an arbitrary predetermined final state  $x(t_1)$ .

System (1) is called **fully controllable in terms of output** if the choice of the control action  $u(t)$  over the time interval  $[t_0, t_1]$  makes it possible to transfer the system from any initial state  $x(t_0)$  to such a final state at which a predetermined arbitrary value of the output  $y(t_1)$  is provided.

A system (1) is called **fully observable** if the initial state  $x(t_0)$  *can be determined* from the reaction  $y(t)$  *at the output of the system, in the interval of time  $[t_0, t_1]$  with a given control action  $u(t)$ .*

Matrix Rank  $A$  is the largest of the orders of minors of the matrix  $A$  other than zero. The rank of the zero matrix is considered to be zero. The rank of a matrix is equal to the maximum number of its rows that form a linearly independent system.

The problem statement is formulated as follows:

Let the matrices  $A$ ,  $B$ ,  $C$ , and system (1) be known. It is necessary to determine whether the system is completely controllable and observable.

### **Criteria of controllability and observability**

**Criterion of controllability by state.** In order for system (1) to be fully controllable by state, it is necessary and sufficient that the rank of the matrix of controllability by state

$$W = (B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B)$$

was equal to the dimension of the state vector:

$$\text{rang}W = n.$$

**Criterion of controllability by output.** In order for system (1) to be fully controllable by output, it is necessary and sufficient that the rank of the matrix of controllability by output

$$P = (CB \quad CAB \quad CA^2B \quad \dots \quad CA^{n-1}B)$$

was equal to the dimension of the output vector

$$\text{rang}P = k.$$

**The criterion of observability.** In order for system (1) to be fully observable, it is necessary and sufficient that the rank of the observability matrix

$$Q = (C^T \quad A^T C^T \quad (A^T)^2 C^T \quad \dots \quad (A^T)^{n-1} C^T)$$

was equal to the dimension of the state vector:

$$\text{rang} Q = n.$$

Algorithm for solving the problem

1. In the equations of state and output, select matrices  $A, B, C$ .
2. Compile the state controllability matrix  $W$ , the output controllability matrix  $P$ , and the observability matrix  $Q$ .
3. Calculate the ranks of the matrices and draw a conclusion about controllability and observability based on the corresponding criterion.

Information from the course of higher mathematics

1. A square matrix of order  $n$  ( $A_{n \times n}$ ) has rank  $n$  if its determinant is non-zero:  $|A| \neq 0$ .
2. The  $T$  sign means the matrix transpose operation. To transpose the matrix, you need to swap the rows and columns.
3. When multiplying matrices, the product matrix has as many rows as the first multiplier matrix and as many columns as the second multiplier matrix. In this case, the number of columns of the first multiplier matrix must coincide with the number of rows of the second multiplier matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ik} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nk} \end{pmatrix}_{n \times k} \cdot \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{k1} & b_{k2} & \dots & b_{kj} & \dots & b_{km} \end{pmatrix}_{k \times m} = \begin{pmatrix} c_{11} & \dots & c_{1m} \\ \dots & c_{ij} & \dots \\ c_{n1} & \dots & c_{nm} \end{pmatrix}_{n \times m}.$$

The element in the product matrix in cell  $(i, j)$  is equal to the dot product of the  $i$ th row vector of the first multiplier matrix and the  $j$ th column vector of the second multiplier matrix:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{ik}b_{kj}.$$

### Gram function

The gram function allows you to compute Gram functions for assessing the controllability and observability of a system, called controllability and observability grams, respectively. Gramians are used to study the controllability and observability properties of system models defined in state space, as well as to construct their minimum implementations. They are more convenient for calculations than controllability and observability matrices (the model matrix A must be stable).

$$G_C = \int_0^{\infty} e^{A\tau} B B^T e^{A^T \tau} d\tau$$

For a continuous ss model, the controllability gramian is defined by the integral, and the observability gramian is defined by the integral

$$G_0 = \int_0^{\infty} e^{A^T \tau} C^T C e^{A\tau} d\tau$$

For discrete models, the analogues of the gramians of controllability and observability are the expressions

$$G_C = \sum_{k=0}^{\infty} A^k B B^T (A^T)^k$$

The gramian of controllability is positively defined if and only if the pair of matrices {A, B} is controllable. Similarly, the gramian of observability is positively defined if and only if the pair of matrices {A, C} is observable.

$$G_0 = \sum_{k=0}^{\infty} (A^T)^k C^T C A^k$$

The `Gc = gram(sys, 'c')` function calculates the manageability gram for a continuous or discrete ss model.

The `Go = gram(sys, 'o')` function calculates the observability gramian for a

continuous or discrete ss model.

The obsv function forms an observability matrix for the model in the state space:

$$Ob = \text{obsv}(A, C)$$

A system is observable if the observability matrix has a full rank. The return value Ob is the observability matrix, which has p rows (p is the number of outputs) and i columns and is described by the relation

$$Ob = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

### **The Canonical Form of Observability**

If the observability matrix for the pair {A, C} has rank  $r < n$ , where n is the order of the matrix A, then there is a similarity transformation of the form

$$\bar{A} = TAT^T, \bar{B} = TB, \bar{C} = CT^T$$

where T is a unitary matrix in which the transformed system has a block-triangular shape with unobservable modes located in the upper left corner:

$$\bar{A} = \begin{bmatrix} A_{no} & A_{12} \\ 0 & A_0 \end{bmatrix}, \bar{B} = \begin{bmatrix} B_{no} \\ B_0 \end{bmatrix}, \bar{C} = [0 \quad C_0]$$

This form is called the canonical form of observability.

The pair of matrices {A,C} is observable, so the relation is valid

$$C_0(pI - A_0)^{-1}B_0 = C(pI - A)^{-1}B,$$

that is, the transfer function of the entire system coincides with the transfer function of its observable part, and all modes corresponding to the eigenvalues of matrix A are unobservable.

The function  $[Abar, Bbar, Cbar, T, k]=ctrbf(A,B,C)$  converts the ss model described by the triple of matrices (A, B, C) into the canonical form of observability  $[Abar, Bbar, Cbar]$ . The ctrbf function forms the so-called canonical form of observability. The matrix T describes the similarity transformation, and the elements of the vector k indicate the number of observable modes highlighted in each step of the transformation matrix calculation. The number of non-zero elements of the vector k indicates how many iterations were required to calculate the matrix T, and the sum value (k) indicates the number of canonical state variables corresponding to the observed part of the matrix Abar. The functions used and their descriptions are presented in the table:

| Features                          | Description                                                                             |
|-----------------------------------|-----------------------------------------------------------------------------------------|
| Ctrb(A,B)                         | Forms a manageability matrix for a model in a state space                               |
| $[Abar, Bbar, Cbar]=Ctrbf(A,B,C)$ | Forms the canonical form of controllability                                             |
| Gram(sys,'c')                     | Calculates the manageability gram for a continuous or discrete ss model                 |
| Gram(sys,'o')                     | Computes the observability gram for a continuous or discrete ss model                   |
| obsv(A, C)                        | Forms an observability matrix for a model in a space-state                              |
| obsvf(A, C)                       | Forms the canonical form of the observability matrix for the model in the spatial state |
| Rank(A)                           | Forms the rank of matrix A                                                              |

### Methodical example

To study the controllability and observability of the system:

$$\begin{aligned}\dot{x}_1 &= -3x_1 + 5x_2 + u, \\ \dot{x}_2 &= 2x_1 + 2u, \quad y = x_1 - x_2\end{aligned}$$

1. In the equations of state and output, let us distinguish the matrices  $A$ ,  $B$ ,  $C$ :

$$A = \begin{pmatrix} -3 & 5 \\ 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad C = (1 \ -1),$$

whose ranks are respectively equal:  $n = 2$ ,  $r = 1$ ,  $k = 1$ .

2. Make matrices of controllability and observability:

$$W = (B \quad AB) = \begin{pmatrix} 1 & 7 \\ 2 & 2 \end{pmatrix},$$

$$P = (CB \quad CAB) = \begin{pmatrix} -1 & 5 \end{pmatrix},$$

$$Q = (C^T \quad A^T C^T) = \begin{pmatrix} 1 & -5 \\ -1 & 5 \end{pmatrix}$$

3. Определяем ранги матриц:  $\text{rank } W = 2 = n$ ,  $\text{rank } P = 1 = k$ ,  $\text{rank } Q = 1 \neq n$ .

According to the criteria of controllability and observability, the system is quite controllable in terms of state and output, but it is not completely observable.

Matrix  $A$  – coefficients for  $x_1$  and  $x_2$  in the equation  $\dot{x}_1$  – line 1, coefficients for  $x_1$  and  $x_2$  in the equation  $\dot{x}_2$  – row 2. Since dimension  $2 \times 2$ , it means  $n = 2$

Matrix  $B$  – coefficients for  $U$  in the equation  $\dot{x}_1$  – line 1, – coefficients for  $U$  in the equation  $\dot{x}_2$  – line 2. Since the dimension is  $1 \times 1$ , it means  $r = 1$ .

Matrix  $C$  is the coefficients at  $x_1$  and  $x_2$  in equation  $Y$ . Since the dimension is  $1 \times 1$ , so  $k = 1$ .

1. Define matrices  $A$ ,  $B$ ,  $C$  in the MatLab environment:

```
>> A=[-3 5;2 0]

A =

    -3     5
     2     0
>> B=[1;2]

B =

     1
     2
>> C=[1 -1]

C =

     1    -1
>>
```

2. Condition controllability criterion. Set in MatLab  $W = (B \ AB \ A^2B \ \dots \ A^{n-1}B)$  and find the rank of the resulting matrix:

```
>> Co=ctrb(A,B)
```

```
Co =
```

```
    1    7
    2    2
```

```
>> rank(Co)
```

```
ans =
```

```
    2
```

3. Output controllability criterion. Set in MatLab  $P = (CB \ CAB \ CA^2B \ \dots \ CA^{n-1}B)$  and find the rank of the resulting matrix:

```
>> [Abar, Bbar, Cbar, T, r]=ctrbf(A,B,C)
```

```
Abar =
```

```
-5.2000    2.4000
-0.6000    2.2000
```

```
Bbar =
```

```
    0.0000
-2.2361
```

```
T =
```

```
-0.8944    0.4472
-0.4472   -0.8944
```

```
Cbar =
```

```
-1.3416    0.4472
```

```
r =
```

```
    1    1
```

Let's look at the dimension  $r$ , it is equal to  $1 \times 1$ , i.e.  $\text{rang } P = 1$ .

3. Observability criterion. Set in MatLab  $Q = (C^T \ A^T C^T \ (A^T)^2 C^T \ \dots \ (A^T)^{n-1} C^T)$  and find the rank of the resulting matrix:

```
>> Ob=obsv(A,C)
ans =
1
```

$\text{Rang } Q \neq n$ , so the system is not observable.

### Variants of tasks.

Determine controllability by state and output, observability of the system using the criteria of controllability and observability, as well as using the Gram function

| No Option | Equations of state and output                                                                  |
|-----------|------------------------------------------------------------------------------------------------|
| 1         | $\dot{x}_1 = x_2$<br>$\dot{x}_2 = u, \quad y = x_1$                                            |
| 2         | $\dot{x}_1 = -5x_1 + 2x_2 + 3u$<br>$\dot{x}_2 = 4x_1 + u, \quad y = 5x_1 - 3x_2$               |
| 3         | $\dot{x}_1 = 7x_1 - x_2 + 2u$<br>$\dot{x}_2 = -7x_1 - 2u, \quad y = -x_1 + 5x_2$               |
| 4         | $\dot{x}_1 = x_1 + x_2 - 5u$<br>$\dot{x}_2 = -7x_1 - 2u, \quad y = -x_1 + 5x_2$                |
| 5         | $\dot{x}_1 = x_2 + u, \quad \dot{x}_2 = -x_1 - 2x_2 - u$<br>$y_1 = x_1, \quad y_2 = x_1 + x_2$ |

|    |                                                                                                                              |
|----|------------------------------------------------------------------------------------------------------------------------------|
| 6  | $\dot{x}_1 = 7x_2 - 3u, \quad \dot{x}_2 = -3x_1 - 4x_2 + 5u$<br>$y_1 = x_2, \quad y_2 = 3x_1 + 2x_2$                         |
| 7  | $\dot{x}_1 = -x_2 + 6u, \quad \dot{x}_2 = 3x_1 + x_2 - 8u$<br>$y_1 = 3x_1 - 5x_2, \quad y_2 = x_1 + 8x_2$                    |
| 8  | $\dot{x}_1 = x_2 - u, \quad \dot{x}_2 = 5x_1 + 2x_3 + u$<br>$\dot{x}_3 = -2x_1 - 2x_3 - u, \quad y = -2x_1 + x_2$            |
| 9  | $\dot{x}_1 = x_1 + 3x_2 - 3u, \quad \dot{x}_2 = 2x_1 - 4x_3 + 2u$<br>$\dot{x}_3 = -5x_1 + 3x_3 - 4u, \quad y = -7x_1 - 2x_2$ |
| 10 | $\dot{x}_1 = -x_1 - 5x_2 + 7u, \quad \dot{x}_2 = x_1 - 2x_3 - 9u$<br>$\dot{x}_3 = x_1 + x_3 - u, \quad y = -6x_1 + x_2$      |

## Linear System Models

Several methods can be used to describe linear systems:

- Differential Equations
- Models in State Space
- Transfer functions
- Zero-Pole Models

The first two methods are called *temporal methods*, since they describe the behavior of the system in the time domain and reflect the internal connections between signals. Transfer functions and zero-pole models belong to *the frequency methods of description*, since they are directly related to the frequency characteristics of the system and reflect only the input-output properties (that is, they do not fully describe the dynamics).

Frequency methods allow algebraic methods to be applied to analysis and synthesis, which often simplifies calculations. On the other hand, methods based on state space models are more suitable for automatic calculations because they use computationally stable linear algebra algorithms.

The initial equations of the dynamics of objects, which are built on the basis of the laws of physics, are in the form of nonlinear differential equations. For approximate analysis and synthesis, they are usually linearized in the vicinity of the steady state and linear differential equations are obtained.

A linear equation  $\ddot{y} + 2\dot{y} + 3y = 4\dot{u} + 5u$  can be written in operator form

$$(p^2 + 2p + 3)y = (4p + 5)u \quad \text{or} \quad D(p)y = N(p)u$$

where  $u(t)$  is the input signal,  $y(t)$  is the output signal,  $p = \frac{d}{dt}$  is the differentiation operator,  $D(p) = p^2 + 2p + 3$  and  $N(p) = 4p + 5$  is the operator polynomials.

**The transfer function**  $W(s)$  of a linear stationary system from a complex variable  $s$  is defined as the ratio of the Laplace transform of the output to the Laplace transform of the input under zero initial conditions

$$W(s) = \frac{Y(s)}{U(s)}, \quad Y(s) = \int_0^{\infty} y(t)e^{-st} dt, \quad U(s) = \int_0^{\infty} u(t)e^{-st} dt.$$

The transfer function of the link, which is described by the above equation, is equal to

$$W(s) = \frac{4s + 5}{s^2 + 2s + 3},$$

that is, coincides with the ratio of operator polynomials  $N(p)/D(p)$  when replacing the variable  $p$  with  $s$ .

In the Matlab ENVIRONMENT, THE TRANSFER FUNCTION is introduced as the ratio of two polynomials (polynomials) from the complex variable  $s$ . Polynomials are stored as arrays of coefficients written in **descending order of powers**. For example, the transfer function

$$F(s) = \frac{2s + 4}{s^3 + 1.5s^2 + 1.5s + 1}$$

is administered as follows<sup>1</sup>

```
>> n = [2 4]
n =
    2    4
>> d = [1 1.5 1.5 1]
d =
    1.0000    1.5000    1.5000    1.0000
>> f = tf ( n, d )
Transfer function:
      2 s + 4
-----
s^3 + 1.5 s^2 + 1.5 s + 1
```

or immediately, without preliminary construction of the numerator and denominator:

```
>> f = tf ( [2 4], [1 1.5 1.5 1] );
```

An object of class `tf` is created in memory that describes the transfer function. A semicolon at the end of the instruction suppresses the output to the screen.

---

<sup>1</sup> Black indicates user input, blue indicates the response of the MATLAB ENVIRONMENT.

From the transfer function, you can easily build a model in the form of "zero-poles"

```
>> f_zpk = zpk(f)
```

Zero/pole/gain:

2 (s+2)

-----

(s+1) (s^2 + 0.5s + 1)

Zeros are the roots of the numerator, poles are the roots of the denominator. This function has one zero at a point  $s = -2$  and three poles at points  $s = -1$  and  $s = -0,25 \pm 0.9682i$ .

The model in the state space is associated with the notation of differential equations in the standard Cauchy form (in the form of a system of first-order equations):

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Here  $x$  is the vector of the variables of the size state  $n \times 1$ ,  $u$  is the vector of input signals (control vector) of the size  $m \times 1$  and  $y$  is the vector of the output signals of the size  $p \times 1$ . In addition,  $A, B, C$  and  $D$  are constant matrices. According to the rules of matrix calculations, the matrix  $A$  must be square in size  $n \times n$ , the matrix  $B$  has the size  $n \times m$ , the matrix  $C - p \times n$  and the matrix  $D - p \times m$ . For systems with one input and one output,<sup>2</sup> the matrix  $D$  is a scalar quantity.

To convert a transfer function to a model in state space, use the

```
>> f_ss = ss ( f )
```

a =

|    | x1   | x2      | x3       |
|----|------|---------|----------|
| x1 | -1.5 | -0.1875 | -0.03125 |
| x2 | 8    | 0       | 0        |
| x3 | 0    | 4       | 0        |

---

<sup>2</sup> In foreign literature, the abbreviation SISO = Single Input Single Output is used for one-dimensional systems.

```

b =
    U1
    x1  0.5
    x2   0
    x3   0
c =
    x1  x2  x3
    y1 0 0.5 0.25
d =
    U1
    y1 0

```

This means that the model matrices have the form

$$A = \begin{bmatrix} -1.5 & -0.1875 & -0.03125 \\ 8 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix}, C = [0 \quad 0.5 \quad 0.25], D = 0.$$

A model in a state space can be constructed not for all transfer functions, but only for *correct ones*, in which the power of the numerator is not higher than the power of the denominator. For example, the transfer function

$$W(s) = \frac{2s^2 + 3s + 1}{s + 5}$$

– incorrect, it cannot be transformed into a model in the space of states.

The concept of a *strictly correct function is also used*, in which the power of the numerator *is less* than the degree of the denominator. If you build a model in the state space for such a function, the matrix  $D$  will be equal to zero, that is, there is no direct transmission from input to output (with a jumpy change in the input, the signal at the output will be continuous).

### Steady-state gain

One of the most important characteristics of a linear system is steady-state gain or *static gain (DC-gain)*. It can be defined as the steady-state value of the output signal with a constant input signal equal to unity.

Consider a differential equation

$$\ddot{y} + 2\dot{y} + 3y = 4\dot{u} + 5u.$$

Assuming that all derivatives (in the steady-state mode) are equal to zero, we get

$$3y = 5u \Rightarrow y = \frac{5}{3}u.$$

Static gain is  $k_s = 5/3$ .

If a transfer function is given, for the calculation it  $k_s$  is necessary to substitute  $s=0$ , since the variable  $s$  corresponds to the differentiation operator. The equation discussed above can be compared to the transfer function

$$W(s) = \frac{4s + 5}{s^2 + 2s + 3}.$$

Then

$$k_s = \lim_{s \rightarrow 0} W(s) = \frac{5}{3}.$$

If the system contains an integrating link (the transfer function has a pole at a point  $s=0$ ), this limit is equal to infinity, that is, with a constant signal, the output infinitely increases or decreases before reaching the steady state.

The same result can be obtained using an equivalent model in the state space. Using THE MATLAB framework , we find

$$A = \begin{bmatrix} -2 & -1.5 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, C = [2 \quad 1.25], D = 0.$$

Assuming  $\dot{x} = 0$ , we obtain a model that determines the steady state

$$\begin{aligned} 0 &= Ax + Bu \\ y &= Cx + Du \end{aligned} \Rightarrow \begin{aligned} x &= -A^{-1}Bu \\ y &= Cx + Du \end{aligned} \Rightarrow y = (-CA^{-1}B + D)u,$$

From where it follows

$$k_s = -CA^{-1}B + D.$$

For our system, as before, we get  $k_s = \frac{5}{3}$ . Note that in order for the static gain to be finite, the reversibility of the matrix is required, that is, the absence of integrating links<sup>3</sup>.

To find the static gain of model **f** in MATLAB, use the

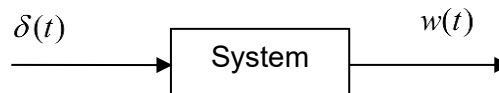
**>> k = dcgain ( f )**

## Impulse Response

The *impulse characteristic* (weight function)  $w(t)$  is the response of a system to a single infinite momentum (delta function or Dirac function) under zero initial conditions. The delta function  $\delta(t)$  is defined by the equations

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}, \quad \int_{-\infty}^{\infty} \delta(t) dt = 1.$$

This is a *generalized function* – a mathematical object that represents an ideal signal, no real device is able to reproduce it. The delta function can be considered as the limit of the rectangular momentum of a unit area centered at a point  $t = 0$  when the width of the pulse tends to zero.



The second name, the *weight function*, is due to the fact that for an arbitrary input signal, the output of the  $u(t)$  system  $y(t)$  is calculated as a convolution

$$y(t) = \int_{-\infty}^t u(\tau) w(t - \tau) d\tau = \int_0^{\infty} u(t - \tau) w(\tau) d\tau.$$

Here, the function  $w(t)$  "weights" the input signal in an integrative expression.

The impulse response reflects only the input-output ratios under *zero initial conditions*, that is, it cannot fully describe the dynamics of the system.

The concept of impulse response is used mainly for systems whose transfer functions *are strictly correct*. If the transfer function is correct, but not strictly correct,

---

<sup>3</sup> The poles of the transfer function are the eigens of the matrix  $A$ . Thus, if the transfer function has a pole at the point  $s = 0$ , the matrix  $A$  will be degenerate.

the direct transfer coefficient from input to output (model matrix  $D$  in state space) is not zero, so the infinite momentum at the input is  $t=0$  transmitted to the output at the moment. Such an (infinite in magnitude) impulse response cannot be constructed. THE MATLAB system in this case plots the impulse response for the strictly correct part, taking  $D=0$ . This is one of those cases when the computer produces a qualitatively incorrect result.

If the system does not contain integrators, the impulse response tends to zero. This follows from the limit value theorem:

$$\lim_{t \rightarrow \infty} w(t) = \lim_{s \rightarrow 0} s W(s),$$

where  $W(s)$  is the transfer function of the system, which is the Laplace transform for  $w(t)$ . The impulse response of a system with one integrator tends to a constant value equal to the static transfer coefficient of the system without an integrator. For a system with two integrators, the impulse response asymptotically tends to a straight line, with three integrators – to a parabola, and so on.

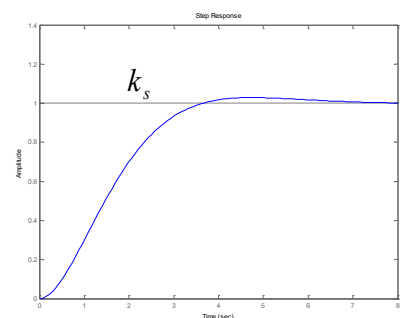
## Transient Response

*The transient characteristic* (transient function)  $h(t)$  is the response of the system (under zero initial conditions) to a single step signal (a single jump)

$$1(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}.$$

The momentum and transient functions are connected by expressions

$$w(t) = \frac{dh(t)}{dt}, \quad h(t) = \int_0^t w(\tau) d\tau.$$

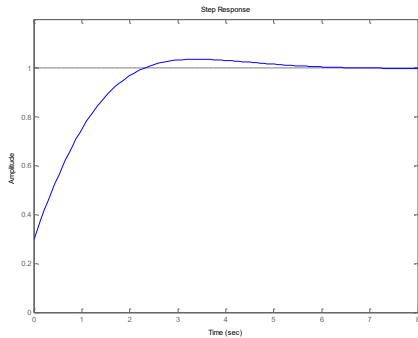


For systems without integrators, the transient characteristic tends to a constant value. The transient characteristic of a system with a differentiating link (the numerator of the transfer function has zero at a point  $s=0$ ) tends to zero. If the system contains integrating links, the transient characteristic asymptotically tends to a line, parabola, etc., depending on the number of integrators.

By definition, the limit value of the transient function  $h(t)$  at  $t \rightarrow \infty$  is the static gain:

$$k_s = \lim_{t \rightarrow \infty} h(t).$$

This value makes sense only for stable systems, since in the case of instability, the transition process does not converge to the final value.



If the transfer function is correct, but not strictly correct (the model matrix  $D$  in the state space is not equal to zero), the jump change in the input signal instantly leads to a jump change in the output. The magnitude of this jump is equal to the ratio of the coefficients at the highest powers of the numerator and the denominator of the transfer function (or the model matrix  $D$  in the state space).

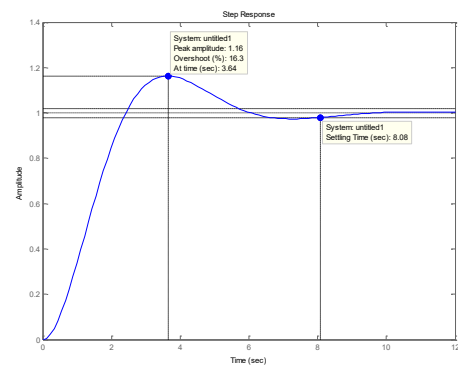
Based on the transient characteristic, you can find the most important indicators of the quality of the system - overshoot and settling time.

*Overregulation* is defined as

$$\sigma = \frac{h_{\max} - h_{\infty}}{h_{\infty}} \times 100\%,$$

where  $h_{\max}$  is the maximum value of the function

$h(t)$ , and is  $h_{\infty} = \lim_{t \rightarrow \infty} h(t)$  the steady-state value of the output.



*Transient time* is the time after which the output signal deviates from the steady-state value by no more than a specified small amount (Matlab uses an accuracy of 2% by default).

## LIST OF BASIC AND ADDITIONAL LITERATURE NECESSARY FOR MASTERING THE DISCIPLINE (MODULE)

### List of basic literature:

Korobko, V. I.; Management Theory Electronic resource: Textbook for university students studying in the specialty "State and Municipal Management",

"Organization Management" / V. I. Korobko. - Management Theory, 2022-03-26. - Moscow : UNITI-DANA, 2017. - 383 p. - The book is in the premium version of EBS IPR BOOKS. - ISBN 978-5-238-01483-8

Tsvetkova, O. L.; Theory of Automatic Control: Textbook / O.L. Tsvetkova. - Moscow|Berlin : Direct-Media, 2016. - 207 p. : ill., schemes, tabl. - <http://biblioclub.ru/>. - Bibliogr. in the book. - ISBN 978-5-4475-8334-7

### **List of additional literature:**

Muzyleva, I. V.; Theory of Automatic Control. Linear Systems Electronic resource: Guidelines for Practical Classes / I. V. Muzyleva, A. A. Muravyov. - Lipetsk : Lipetsk State Technical University, EBS ASV, 2013. - 84 p. - The book is in the premium version of EBS IPR BOOKS. - ISBN 978-5-88247-613-6

8.2. List of educational and methodological support for students' independent work in the discipline (module)

Zyabrov, V. A.; Fundamentals of Automation and Theory of Control of Technical Systems Electronic resource: Methodological recommendations / V. A. Zyabrov, D. A. Popov. - Fundamentals of Automation and Control Theory of Technical Systems, 2019-06-22. - Moscow: Moscow State Academy of Water Transport, 2015. - 46 p. - The book is in the premium version of EBS IPR BOOKS. - ISBN 2227-8397

Semenov, A. M.; Fundamentals of Control Theory. Linear Systems Elektronnyye resurs : Uchebno-metodicheskoe posobie po laboratornomu praktikumu [Electronic resource] : Uchebno-metodicheskoe posobie po laboratornomu praktikumu [Educational and methodological manual on laboratory practice] / A. M. Semenov, V. V. Panichev. - Orenburg : Orenburg State University, EBS ASV, 2017. - 181 p. - The book is in the premium version of EBS IPR BOOKS. - ISBN 978-5-7410-1842-2