

GUIDELINES

By completion of practical exercises

By discipline

« Digital processing signals /Digital Signal Processing»

for students in the field of study

11.03.02 Infocommunication technologies And systems connections Direction (profile)

"Intellectual processing data V infocommunication systems and networks"

Practical lesson No. 1 HARMONIC ANALYSIS OF PERIODIC SIGNALS

1.1. Target work

At present, when forming a mathematical model of a signal, the most widely used system is one in which the basis is formed by orthogonal harmonics (sinusoidal and cosine) functions. This is explained those, What harmonic function is the only one that retains its shape when passing through through linear electrical chain. Performance *an arbitrary* signal in the form of a sum of harmonic oscillations is called *spectral decomposition* of this signal in the basis of harmonic functions, or *harmonic analysis* of the signal.

The aim of the work is to determine the coefficients of the Fourier series and construct an approximating function for a periodic signal formed from pulses of a given shape.

1.2.

1.3. Main concepts And settlement formulas

Let the signal under study be described by a periodic function of time $x(t)$, which satisfies the Dirichlet conditions within the period of its change T . Then the function $x(t)$ can be represented as a Fourier series.

Like this in this way, V in general case periodic signal $x(t)$ content - lives V to myself independent constant from time to time component $a_0/2$. an infinite set of harmonic oscillations with frequencies $\omega_n = n2\pi/T$ ($n=1,2,3,\dots$), multiples *basic* frequency $\omega_1 = 2\pi/T$ periodic signal. Aggregates coefficients a_n, b_n ($n=1,2,\dots$) decomposition periodic functions $x(t)$ V Fourier series they call frequency spectra of this function.

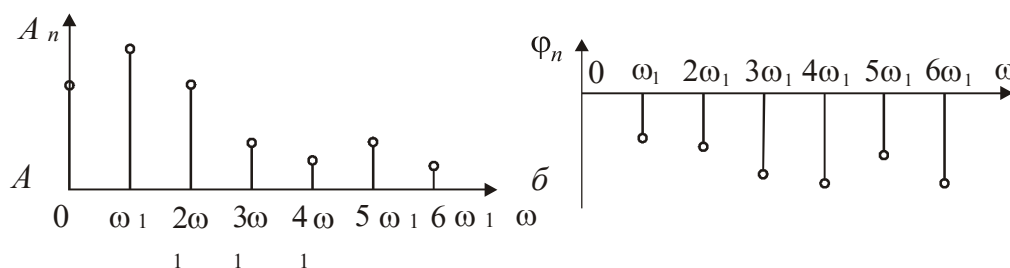
Widespread And other form records rows Fourier:

Harmonic component
they call *basic harmonica*, A harmonic component
 $x(t)$

- n -th harmonica.

Sets of quantities A_n, φ_n ($n=1,2,\dots$) they call accordingly
And

but *the amplitude and phase* frequency spectra of the signal or, in other words, spectrum of amplitudes and spectrum phases. Frequency spectra are functions that depend on the harmonic number n as an independent variable. exchange. Graphically frequency spectra depict V view segments A_n And φ_n , conducted perpendicularly To axes, on which are applied know - ching $\omega = n\omega_1$ (rice. 1.1). Graphic image amplitude And F- callous frequency spectra accepted call *amplitude And phase - howl spectral diagrams*.



Rice. 1.1. Amplitude And phase spectral diagrams

Row Fourier provides the best V meaning root mean square errors approximation To original functions. This means, What

if the number of terms of the Fourier series is limited and the original signal $x(t)$ approximated by a function

That the smallest average quadratic error

$$\sigma^2 = \int_{t_0}^{t_0+T} [x(t) - \tilde{x}(t)]^2 dt \quad (1.10)$$

occurs in the case when the coefficients of the Fourier series are determined by formulas (1.2) – (1.4).

As the number of terms in the Fourier series increases to infinity, the mean square error (1.10) tends to zero. However, this is not at all means, What row exactly strives To functions at any meaning time t . For example, If function $x(t)$ has gap V point t_1 , That There is

$$x(t_1 - 0) \neq x(t_1 + 0), \quad (1.11)$$

then the Fourier series at this point, according to Dirichlet's theorem, converges to the arithmetic mean:

Like this in this way, convergence rows Fourier in in many ways depends from Ana -

lytic properties of the expanded function. A smoother function $x(t)$ corresponds to better convergence of its Fourier series.

Average power periodic signal

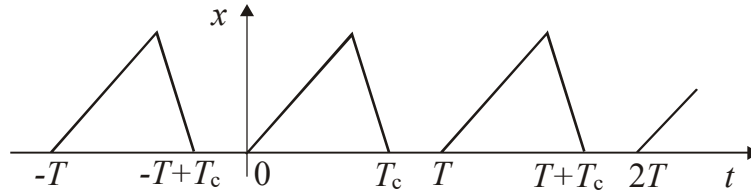
$$P_{sr} = \frac{1}{T} \int_0^T x^2(t) dt, \quad (1.13)$$

If the amplitude spectrum of the signal is known, it can be calculated, according to Parseval's theorem, using the formula

For a signal described by a truncated Fourier series, one can find an approximate value of the average power

1.4. Methodological instructions

In this paper, a harmonic analysis of a periodic signal formed from pulses of a given shape is carried out (see Appendix P.1). In Fig. 1.2 shows an example of a graph of a signal obtained from triangular pulses.



Rice. 1.2. Example periodic signal

At calculation coefficients rows Fourier required ask boss - ny moment t_0 period integration V formulas (1.2) - (1.4). How rule, meaning t_0 choose from the condition simplifications calculations. For studied V work signals we recommend accept $t_0 = 0$. At this the lower and upper limits of integration in formulas (1.2) – (1.4) will be equal to 0 and T_c , respectively.

We appeal attention on That, What standard function $\arctg x$ V sis - MatchCAD programming theme defines the main values, limited by the limits :

$$-\frac{\pi}{2} \leq \arctg x \leq \frac{\pi}{2}.$$

Therefore, if we use the formula recommended in the literature when determining the harmonic phase ,

$$\varphi_n = \arg (a_n - j b_n) = - \arctg \frac{b_n}{a_n},$$

Can get unfaithful results.

To get correct results at use func - tions $\arctg x$, the signs of the coefficients should be taken into account a_n, b_n and make the necessary adjustments to the calculation formulas.

At execution work we offer use existing in the MatchCAD system function

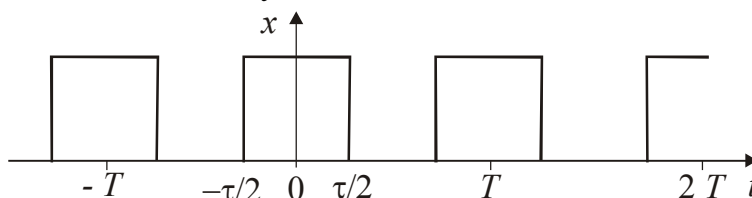
$$\varphi_n = \arg (a_n - j b_n),$$

which defines values φ_n from $-\pi$ to π .

1.5. Program work

1.5.1. The main thing exercise

1. Obtain analytical expressions for the coefficients a_0, a_n, b_n decomposition V row Fourier periodic sequences rectangular impulses (rice. 1.3). Calculate values A_n, φ_n per-out five harmonics For arbitrary τ, T .



Rice. 1.3. Subsequence rectangular impulses

2. Form V environment MathCAD mathematical model given-leg periodic signal $x(t)$. Build schedule signal.

Note : The shape of the signal being studied and the values of the signal parameters are specified by the teacher in accordance with Appendix P.1 and Table 1.1.

Table 1.1

Param eters	N o m f r a n d c a r a n t o c							
	1	2	3	4	5	6	7	8
A, B	1	2	5	10	20	25	40	50
τ, ms	0.3	0.5	0.4	0.5	0.8	1.2	0.8	1.2
T_s, ms	0.4	0.7	0.6	0.8	1.0	1.5	1.0	1.5
T, ms	0.5	0.8	1.0	1.2	1.5	2.0	2.4	3.0

3. Compose program For calculation coefficients a_n, b_n, A_n, φ_n ($n = 0, 1, \dots, 10$) Calculate the values of these coefficients and enter them into the table. Based on the obtained data, construct amplitude and phase spectral diagrams.

4. To educate approximating signal $x^*(t)$ by N way summirova - nia constant component And given numbers first harmonics. By-build on one drawing graphs original $x(t)$ And I approximate-shch $x_N^*(t)$ signals For $N = 5, N = 10$ And $N = 20$. Draw conclusions.

5. Get signals mistakes $\varepsilon(t) = x(t) - x_N^*(t)$ For $N = 5, N = 10$ And $N = 20$. Build their graphics.

6. By formula (1.10) calculate values average quadratic mistakes For $N = 5$, $N = 10$ And $N = 20$. Do conclusions O influence quantity - the number of harmonics taken into account by the value of the mean square error.
7. Using formula (1.13), calculate the average power of the signal being studied. Using formula (1.15), calculate the average power of the periodic - sky signal, described truncated near Fourier, for $N = 5$, $N = 10$ And $N = 20$. Do conclusions O influence quantities taken into account harmonics on size average power signal.

1.6. Control questions And tasks

1. Describe main properties periodic signals.
2. Draw the graphs of the first few basis functions of the Fourier series.
3. Explain the features of the expansion of odd and even functions in Fourier series.
4. Write it down formulas, connecting coefficients trigonometric and complex Fourier series.
5. How will be reflected change provisions countdown time $t = 0$ pe-periodic signal on values A_n And φ_n rows Fourier?
6. How will change spectrum periodic signal, If change the scale along the time axis?
7. How will change spectrum sequences rectangular impulses , If decrease duration τ And period T impulses V two times?
8. Why can't discontinuous functions have an exact approximation by Fourier series?
9. IN how consists of effect Gibbs?

Practical lesson No. 2 HARMONIC ANALYSIS OF NON-PERIODIC SIGNALS

2.1. Target work

Most signals being processed are non - periodic . A feature of harmonic analysis of non - periodic signals is that the relationship between the time function $x(t)$ and her image $X(j\omega)$ in the frequency domain is determined by the integral

ratios that make up couple transformations Fourier.

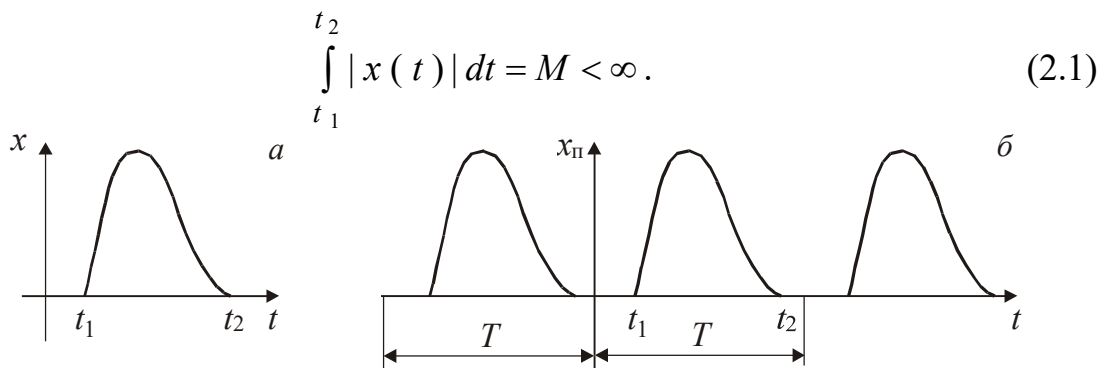
the direct and inverse transformations Fourier And acquisition practical skills their use

For calculation of spectral characteristics $X(j\omega)$ signal $x(t)$ And uprising -

updates functions $x(t)$ By spectral characteristics $X(j\omega)$.

2.2. Main concepts And settlement formulas

Let non-periodic signal is described function time $x(t)$, given on interval (t_1, t_2) (rice. 2.1, a). For functions performing - etsya condition absolute integrability:



Rice. 2.1. Education auxiliary periodic functions:
A - non-periodic function; b - periodic function

By way of repetitions functions $x(t)$ With period $T > t_2 - t_1$ we form auxiliary periodic function

$$x_{\Pi}(t) = \sum_{k=-\infty}^{\infty} x(t - kT). \quad (2.2)$$

Fragment functions $x_p(t)$ shown on rice. 2.1, b . Obviously, What

$$x(t) = \lim_{T \rightarrow \infty} x_p(t)$$

$$x_p(t). \tag{2.3}$$

Periodic function in complex form:

Can describe With help rows Fourier

$$x_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_1 t} \quad (2.4)$$

Where $\omega_1 = 2\pi/T$, A coefficients c_n are calculated By formula

$$c_n = \frac{1}{T} \int_{t_1}^{t_2} x_p(t) e^{-jn\omega_1 t} dt. \quad (2.5)$$

Having substituted (2.5) in (2.4) And replacing $T = 2\pi/\omega_1$, we will receive

$$x_p(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{t_1}^{t_2} x_p(\tau) e^{-jn\omega_1 \tau} d\tau e^{jn\omega_1 t}. \quad (2.6)$$

In limit at $T \rightarrow \infty$ corner frequency $\omega_1 = 2\pi/T$ turns into endlessly small increment frequencies $d\omega$, frequency n -th I make up-cabbage soup rows $n\omega_1$ - V current frequency ω , A operation summation transition - dit V operation integration. At this distance between spectrominesweepers lines, equal basic frequency ω_1 , becomes infinitely - But small, A spectrum - solid.

Like this in this way, at $T \rightarrow \infty$ from formulas (2.6) we will have

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} \left[\int_{t_1}^{t_2} x(\tau) e^{-j\omega \tau} d\tau \right] d\omega. \quad (2.7)$$

WITH taking into account, What values t_1 and t_2 Not defined, let's introduce designation

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt. \quad (2.8)$$

Then

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega. \quad (2.9)$$

Formulas (2.8) And (2.9) are installing unambiguous correspondence between presentation $x(t)$ signal in temporal areas And his pre-

by setting $X(j\omega)$ V areas frequencies. Formula (2.8) carries out straight-
 my transformation And allows find spectral characteristics
 $X(j\omega)$, corresponding to the signal $x(t)$ Symbolically, this is written as
 follows:

$$X(j\omega) = \mathfrak{F}\{x(t)\}.$$

At famous spectral characteristics $X(j\omega)$ By formula (2.9) is being carried out the opposite transformation And is calculated instant meaning signal $x(t)$. Symbolically This Can write down So:

$$x(t) = \mathfrak{F}^{-1} \{ X(j\omega) \}.$$

It has been established, What signal $x(t)$ Can compare his spectral - nuyu characteristics $X(j\omega)$ V volume case, If this signal is described absolutely integrable function, T. e. exists integral

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty.$$

This condition significantly reduces the class of acceptable signals. However, there are mathematical techniques that can be used to obtain spectral characteristics non-integrable signals. These spectral characteristics are generalized functions.

Spectral characteristic $X(j\omega)$ signal $x(t)$, having used famous formula Euler, Can write down V next view:

$$\begin{aligned} X(j\omega) &= X(\omega) e^{j\varphi(\omega)} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \\ &= \int_{-\infty}^{\infty} x(t) \cos \omega t \cdot dt - j \int_{-\infty}^{\infty} x(t) \sin \omega t \cdot dt = a(\omega) - jb(\omega). \end{aligned} \quad (2.10)$$

Valid Part

$$a(\omega) = \int_{-\infty}^{\infty} x(t) \cos \omega t \cdot dt \quad (2.11)$$

spectral characteristic is an even function of frequency, and the imaginary part

$$b(\omega) = \int_{-\infty}^{\infty} x(t) \sin \omega t \cdot dt \quad (2.12)$$

– an odd function of frequency. It follows that the modulus of the spectral characteristic

$$\begin{aligned} |X(\omega)| &= |X(j\omega)| \sqrt{a^2(\omega) + b^2(\omega)} \\ &= \end{aligned} \quad (2.13)$$

is an even function frequencies, A argument spectral characteristics - sticks

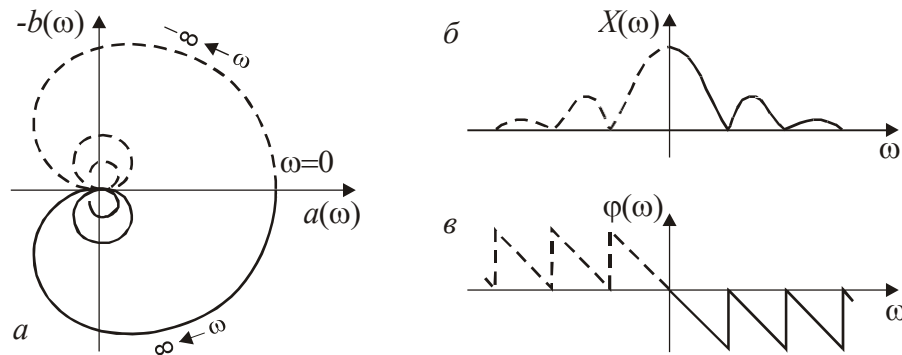
$$\varphi(\omega) = \arg X(j\omega) = \arg [a + jb(\omega)] \quad (2.14)$$

- odd function frequencies.

Spectral characteristics $X(j\omega)$ can be depicted on complex planes via view hodograph (rice. 2.2, *a*). More often same spectral - nuyu characteristics $X(j\omega)$ represent V view amplitude- frequency $X(\omega)$ and phase -frequency $\varphi(\omega)$ spectral characteristics (rice. 2.2, *b*, *c*). Taking into account the symmetry of the spectral characteristics for positive and negative frequency values ω , as a rule, they are constructed only in the range of positive frequency values ω .

Formula (2.9) for the inverse Fourier transform involves the integration of complex functions and is therefore not always convenient for direct calculations. Using Euler's formula and expression (2.10), the formula for the inverse transform can be reduced to the following form:

$$x(t) = \frac{1}{\pi} \int_0^{\infty} [a(\omega) \cos \omega t + b(\omega) \sin \omega t] d\omega. \quad (2.15)$$



Rice. 2.2. Spectral characteristics:
A - hodograph; b - amplitude; V - phase

2.3. Methodological instructions

Program work provides for definition spectral characteristics given signal $x(t)$ With help formulas direct pre-Fourier transformations and signal reconstruction based on the available spectral characteristics using the inverse Fourier transform formula.

Proposed For research signal represents by yourself ody - night pulse, given on interval time $[0, T_s]$ (cm. appendix - (P.1). Its spectral characteristic is calculated using formula (2.8), or using formulas (2.11), (2.12), in which the lower and upper limits integration accepted respectively equal 0 and T with . To achieve necessary accuracy calculations spectral character -

istics on interval $[0, T_s]$ definitions signal is taken Not less 50 counts given signal. Right border frequency interval $[0, \omega_s]$, on which the spectral characteristic is calculated, is selected experimentally, taking into account what is said below.

For recovery signal $x(t)$ By his spectral characteristics - stick $X(j\omega)$ is used formula (2.15) reverse transformations Fourier. How it is known, spectral characteristic signal, given on limited interval time $[0, T_s]$, defined on infinite interval frequencies $(-\infty, \infty)$ (or With taking into account symmetry on semi-besce - Nechnom interval $[0, \infty)$. That's why accurate recovery given signal according to formula (2.15) requires integration over a semi - infinite interval $[0, \infty)$. On practice integration carry out on limited interval $[0, \omega_s]$, as a result what appears error recovery. Right border ω_s frequency interval And number That- The checks in which the values of the spectral characteristics are calculated are selected in such a way that all of its features are taken into account.

2.4. Program work

2.4.1. The main thing exercise

1. Form V environment MathCAD mathematical model $x(t)$ signal, representing by yourself pulse given forms.

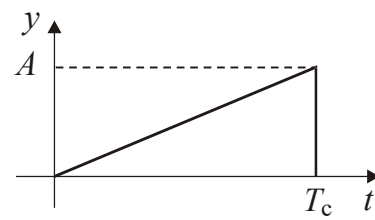
Note: The shape and parameters of the pulse are set by the teacher or selected according to the specified option from Appendix P.1 and Table. 1.1. Parameter α in options 6, 7 the student selects independently .

2. Compose program calculations spectral characteristics $X(j\omega)$ this signal $x(t)$ By formulas direct transformations Ugh-rye . Choose interval $[0, \omega_s]$. Build amplitude $X(\omega)$ And phase - vuyu $\varphi(\omega)$ spectral characteristics.

3. Compose program recovery signal $x(t)$ By received- Noah spectral characteristics $X(j\omega)$ With help formulas back-go transformations Fourier. Build graphs original $x(t)$ And uprising - renewed $x_{in}(t)$ signals. Compare their between by yourself And do quality - military conclusions By results recovery.

4. Change the integration interval $[0, \omega_s]$ and repeat the recovery procedure . Compare with the results obtained in step 3.

5. For signal $y(t)$, the graph of which is shown in Fig. 2.3, according to formula (2.8) we obtain analytical expression for spectral characteristics $Y(j\omega)$. Write analytical expressions for the functions $a(\omega)$, $b(\omega)$.



Rice. 2.3

Note : The pulse parameters are set directly by the teacher or according to the options from Table 1.1.

6. Compose program calculations spectral characteristics $Y(j\omega)$. Construct the amplitude and phase spectral characteristics - sticks researched signal.

2.5. Control questions And tasks

1. Please explain difference between concepts rows And integral Fourier.
2. At what kind conditions Can to use formula direct Fourier transforms?
3. Fair whether principle superpositions For transformations Ugh-rye ?
4. Dana function $x(t) = \delta(t + \tau) + \delta(t - \tau)$. Find spectral characteristics $X(j\omega)$.
5. Please explain difference between one-sided And bilateral Fourier transforms .
6. Name it peculiarities spectral characteristics signals described by odd and even functions.
7. What is the relationship between the spectrum of a single pulse and the spectrum of a periodic signal formed from such pulses?
8. How are changing amplitude And phase spectral characteristics of the signal when it is delayed ?
9. What is happening with spectral characteristic at compression (stretching) signal?
10. How at help transformations Fourier calculate signal energy ?

Practical lesson No. 3 **DECOMPOSITION SIGNALS** **BY SYSTEM FUNCTIONS LAGUERRA**

3.1. Target work

Among the wide variety of orthogonal function systems used, the Laguerre function system occupies a prominent place. This is explained by the fact that Laguerre functions possess a number of advantages. They have found application in problems of forming mathematical models of industrial control objects based on time characteristics obtained experimentally.

The aim of the work is to study the system of Laguerre functions, approximation of a given signal using Laguerre functions and study the influence of the number of terms of the series on the quality of the approximation.

3.2. Main concepts And settlement formulas

Laguerre functions are formed using orthogonal polynomials, the calculation formula for which has the form

$$L_n(\tau) = \frac{e^\tau}{n!} \cdot \frac{d^n}{d\tau^n} \left(\tau^n e^{-\tau} \right), \quad \tau \geq 0. \quad (3.1)$$

The first ones five polynomials, formed By this formula, are described by the following expressions:

$$\begin{aligned} L_0(\tau) &= 1; \\ L_1(\tau) &= 1 - \tau; \\ L_2(\tau) &= 1 - 2\tau + \tau^2 \end{aligned} \quad (3.2)$$

/2;

$$L_3(\tau) = 1 - 3\tau + 3\tau^2/2 - \tau^3/6;$$

$$L_4(\tau) = 1 - 4\tau + 3\tau^2 - 2\tau^3/3 + \tau^4/24.$$

Laguerre polynomials are orthogonal on the semi-axis $[0, \infty)$ with weight

$$\rho(\tau) = e^{-\tau}, \text{ That There is They satisfy condition } \int_0^\infty e^{-\tau} L_n(\tau) L_m(\tau) d\tau = \begin{cases} r_n & \text{при } n = m, \\ 0 & \text{при } n \neq m \end{cases} \quad (3.3)$$

So How polynomials Laguerre form system divergent at

$\tau \rightarrow \infty$ functions, more convenient to use functions Laguerre

$$l_n(\tau) = e^{-\tau/2} L_n(\tau). \quad (3.4)$$

After replacements $\tau = 2\alpha t$ And multiplication on normalizing coefficient -
 (3.4) αt

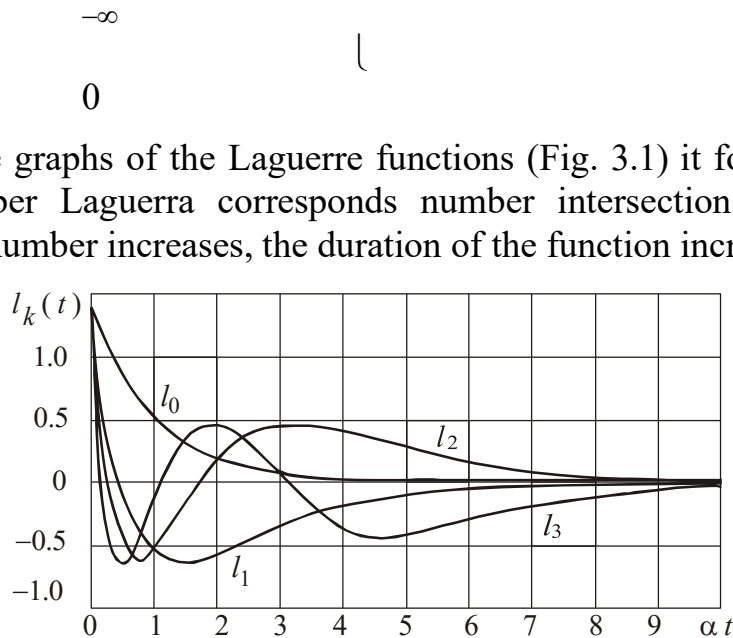
coefficient $\sqrt{2\alpha}$ functions Laguerre accept next view:

$$\begin{aligned}
 l_0(t) &= \sqrt{2\alpha} \cdot \exp(-\alpha t); \\
 &= \sqrt{2\alpha} \cdot \exp(-\alpha t) \cdot (1 - 2\alpha t); \\
 l_1(t) &= \sqrt{2\alpha} \cdot \exp(-\alpha t) \cdot (1 - 4\alpha t + 2\alpha^2 t^2); \\
 &= \\
 l_2(t) &= \sqrt{2\alpha} \cdot \exp(-\alpha t) \cdot (1 - 6\alpha t + 6\alpha^2 t^2 - 4\alpha^3 t^3); \\
 &= \\
 l_3(t) &= \sqrt{2\alpha} \cdot \exp(-\alpha t) \cdot (1 - 8\alpha t + 12\alpha^2 t^2 - 16\alpha^3 t^3 + 8\alpha^4 t^4); \\
 &= \\
 l_4(t) &= \sqrt{2\alpha} \cdot \exp(-\alpha t) \cdot (1 - 10\alpha t + 15\alpha^2 t^2 - 20\alpha^3 t^3 + 10\alpha^4 t^4 - \alpha^5 t^5); \\
 &= \\
 l_n(t) &= \sqrt{2\alpha} \cdot \exp(-\alpha t) \cdot \sum_{j=0}^n (-1)^j \cdot \frac{C_n^j}{j!} \cdot (2\alpha t)^j.
 \end{aligned} \tag{3.5}$$

Here α - large-scale coefficient, C_n^j - number combinations from n By j .

Functions Laguerre form complete system orthonormal functions. For them, the condition is satisfied

$$\int_0^{\infty} l_n(t) l_m(t) dt = \delta_{nm} \quad \text{at} \quad \begin{matrix} n = m, \\ n \neq m. \end{matrix} \tag{3.6}$$



Rice. 3.1. Charts first four functions Laguerre

Decomposition signal By system functions Laguerre is being recorded in the form of the next row

$$x(t) = \sum_{n=0}^{\infty} c_n l_n(t), \quad (3.7)$$

Where

$$c_n = \int_0^{\infty} x(t) l_n(t) dt \quad (3.8)$$

IN practical tasks signal $x(t)$ is being asked on limited interval $[0, T]$ And number taken into account members rows Certainly. Then function $x(t)$ is approximated near

$$x^*(t) \approx \sum_{n=0}^{N-1} c_n l_n(t), \quad (3.9)$$

where

$$c_n = \int_0^T x(t) l_n(t) dt, \quad n=0,1,\dots,N-1. \quad (3.10)$$

Shown, What energy signal, presented V view rows (3.7), is determined expression

$$E_x = \int_0^\infty x^2(t) dt = \sum_{i=0}^\infty c_i^2. \quad (3.11)$$

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$$E^*_{xN} = \sum_{i=0}^{N-1} c_i^2 \quad (3.12)$$

determines the portion of the signal energy that falls on the first N terms of the expansion.

Theoretical studies and the results of practical application have shown that the required accuracy of approximation of weakly oscillatory signals by the truncated series (3.9) is achieved for $N = 4-7$. With a larger number of functions in the truncated series, computational difficulties arise, which are caused, firstly, by the need to calculate power series, and secondly, by the fact that calculating Laguerre functions with large numbers requires a small sampling interval in the initial sections.

3.3. Methodological instructions

Laguerre functions are often used to approximate time series characteristics dynamic systems. TO him, V in particular, refers pulsed transitional function (IPF) $w(t)$, which represents so-fight reaction systems on input signal V view δ -functions.

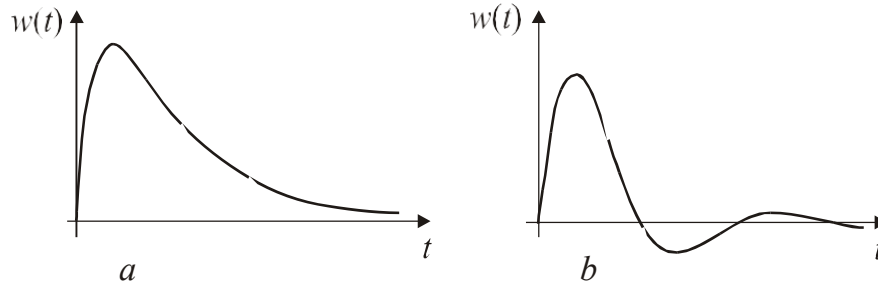
IN this work V quality object research IPF is used systems second order. IPF systems, having two substance - veins poles $p_1 = -a$, $p_2 = -b$ ($a > 0, b > 0$), is determined formula

$$w(t) = \frac{1}{b-a} e^{-at} + \frac{1}{a-b} e^{-bt}. \quad (3.13)$$

IPF systems, which has couple comprehensively conjugated poles $p_{1,2} = -a \pm jb$ ($a > 0, b > 0$), is described expression

$$w(t) = e^{-at} \sin bt. \quad (3.14)$$

Charts functions shown on rice. 3.2.



Rice. 3.2. Pulse transitional functions systems second order:
A - With material poles; *b* - With complex conjugate poles

Interval modeling signal $w(t)$ is selected So, to for outside this interval the signal value did not exceed 0.05 % of the maximum value. At least 50 discrete values of the function are selected from this interval.

Laguerre functions to solve practical problems is the tasks are choice scale values coefficient - α . Usually this value is selected based on the available a priori information about the signal being approximated, and then, after receiving and analyzing the decomposition results, it is refined.

In this paper, it is recommended to select the value of the coefficient α so that the durations of the signal under study and the Laguerre functions used for approximation are approximately equal. Based on this, for the IPF described by expression (3.13), we obtain

$$\alpha \approx 2 \cdot \min(a, b), \quad (3.15)$$

For IPF, described expression (3.14), we will have

$$\alpha \approx 2a. \quad (3.16)$$

Meaning coefficient α Can specify directly By graphs approximated functions $w(t)$ And functions Laguerre.

3.4. Program work

3.4.1. The main thing exercise

1. Compose program modeling given signal

$$w(t).$$

Build schedule functions $w(t)$. Choose meaning α .

Note: Values parameters signal $w(t)$ are being asked prep-giver or are determined according to given option from table 3.1.

Table 3.1

Parameters	Numbers variants							
	1	2	3	4	5	6	7	8
a and b	2	10	6	10	2.5	15	30	5
b	4	25	15	15	4	20	50	10
IPF	$w(t) = \frac{1}{b-a} e^{-at} + \frac{1}{a-b} e^{-bt}$				$w(t) = e^{-at} \sin bt$			

2. Create a program to simulate the first five Laguerre functions. Plot their graphs for the value α , found above. Check the correctness of the value selection α .

Note: Laguerre function graphs are constructed to check the correctness of a set of mathematical expressions; they may not be included in the report.

3. Calculate values integral

$$\int_0^T l_n(t) l_m(t) dt$$

For several arbitrary values n, m ($n, m = 0.4$). Explain the result obtained.

4. Compose program for calculation of values coefficients of the series. Calculate values coefficients $c_n, n = 0.4$. Build spectral diagram.

5. Compose program calculations/functions

$$w^*(t) = \sum_{n=0}^N c_n L_n(t)$$

at $N = 3, 4, 5$. Compare graphs functions $w^*(t)_N$ ($N = 3, 4, 5$) With graph-with a given IPF, plotting them in the same coordinate axes. Draw conclusions.

6. Build graphs mistakes approximations

$$\varepsilon_N(t) = w(t) - w_N^*(t)$$

at $N = 3, 4, 5$. Do analysis their characteristic signs.

7. Compose program For calculation parts energy, coming on first N members decomposition functions $w(t)$. Calculate her values at $N=3,4,5$.

8. Calculate values average quadratic mistakes

$$\sigma^2 = \int_0^T [w(t) - w^*(t)]^2 dt$$

For $N=3,4,5$. Do conclusions.

9. Research influence large-scale coefficient α on size mistakes approximations $\varepsilon_N(t)$ at $N=5$.

3.5. Control questions And tasks

1. Why at decomposition signals use orthogonal functions?
2. Explain the main features of functions that can be represented as expansions in terms of Laguerre functions.
3. Which criterion approximations leads To formula (3.8) to calculate the coefficients of the series?
4. WITH Which purpose V functions Laguerra is introduced multiplier ?
5. Which one? role performs parameter α functions Laguerra ?
6. Explain the relations (3.15) and (3.16) used to determine the value α .
7. What by yourself represents And Which view has spectral diagram of the signal expansion according to the system of Laguerre functions ?
8. Write it down image By Laplace functions Laguerra $l_n(t)$.
9. Please specify main differences orthogonal functions Laguerra and Walsh.

Practical lesson No. 4

DECOMPOSITION SIGNALS BY SYSTEM FUNCTIONS WALSH

4.1. Target work

With the development of computing technology, piecewise constant functions with constant values over fixed time intervals have become more commonly used as basis functions. When calculating the coefficients of a series, the function is multiplied by a constant factor that is unique for a given interval. This is simpler than multiplying by values of continuous basis functions that vary from point to point. The main functions of this type are the Walsh and Haar functions. The Walsh functions, which are the subject of this paper, are used at processing speech signals, signals in biology and medicine, digital image processing, digital holography and many other areas.

The aim of the work is to study the system of orthogonal Walsh functions, decomposition of a signal of a given shape and study the influence of the number of terms of the series on the approximation error.

4.2. Main concepts And settlement formulas

Walsh functions (J. Walsh) were developed in 1923. How development of the Rademacher function system, known at that time, by adding new functions to it. Rademacher functions are formed from sinusoidal functions:

$$rad_0(i) \equiv 1; \quad rad(i) = \text{sign}[\sin(2^i \theta)] \quad I=1, \dots, n \quad (4.1)$$

(plural) ?

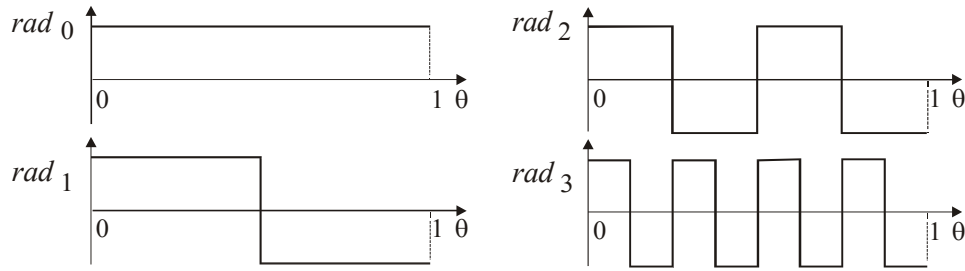
Where θ – relative time changing in the interval $[0, 1)$. Symbol - crowbar

sign (signum function) denotes a function

$$\text{sign}(x) = \begin{cases} 1 & \text{at } x > 0, \\ -1 & \text{at } x < 0. \end{cases} \quad (4.2)$$

According to formulas (4.1) and (4.2), the Rademacher functions take the values +1 or -1 and have the form shown in Fig. 4.1.

Functions Rademacher are orthogonal, But Not make up complete system. On volume same interval $[0, 1)$ exist other functions related by orthogonality conditions to the Rademacher functions. Therefore, the Rademacher function system cannot be effectively used for the expansion of arbitrarily given functions.



Rice. 4.1. Functions Rademacher

Walsh functions are formed from Rademacher functions using the following relation:

$$wal_n(\theta) = \prod_{k=1}^m [rad_k(\theta)]^{n_k}, \quad n = 0, 1, \dots, N-1, \quad (4.3)$$

Where n - number functions Walsh, n_k - meaning (0 or 1) k -th discharge numbers Walsh functions n , recorded V view m -discharge binary code Gray. From here easily see, What quantity functions V system Walsh turns out equal $N = 2^m$, Where m - integer.

The sequence of formation of Walsh functions can be as follows : First, the code of the number n of the Walsh function is written in binary code. Then this number seems to be V code Gray. Code Gray connected With ordinary binary code as follows. If in the ordinary binary number system for a given number we have

$$n = a_{m-1} a_{m-2} \dots a_1 a_0, \quad (4.4)$$

That V code Gray This number is being recorded V view

$$n = b_{m-1} b_{m-2} \dots b_1 b_0, \quad (4.5)$$

где

$$b_0 = a_0 \oplus a_1, \quad b_1 = a_1 \oplus a_2, \quad \dots, \quad b_{m-2} = a_{m-2} \oplus a_{m-1}, \quad b_{m-1} = a_{m-1};$$

Here $\oplus$ – the sign of summation modulo 2. When summing modulo 2 we have:

$$0 \oplus 0 = 0; \quad 0 \oplus 1 = 1; \quad 1 \oplus 0 = 1; \quad 1 \oplus 1 = 0.$$

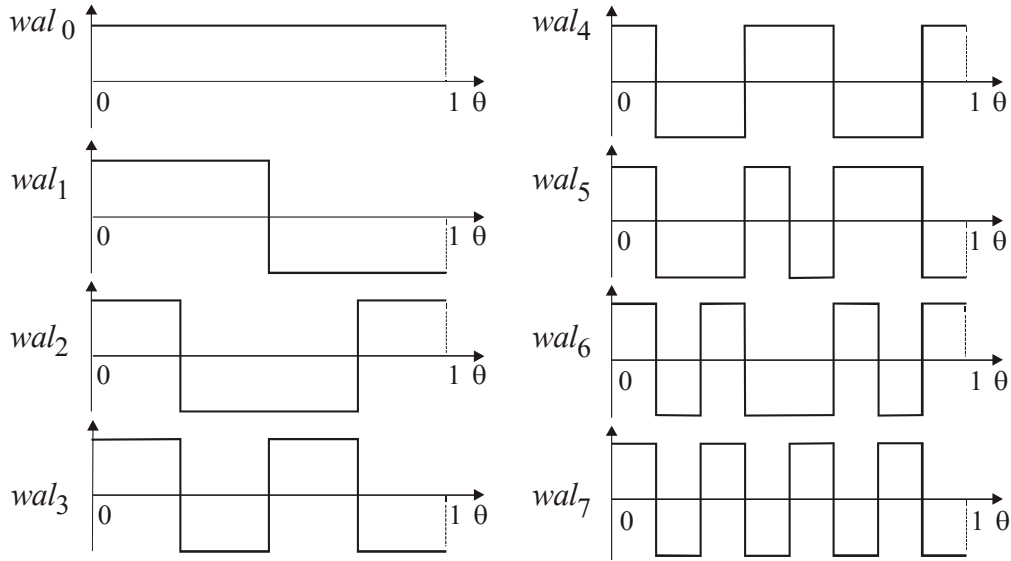
For example, let $n = 4$. IN binary code we will have $n = 100$, V code Gray - $n = 110$. Function Walsh will be recorded So:

$$wal_4(i) = [rad_3(i)]^1 \cdot [rad_2(i)]^1 \cdot [rad_1(i)]^0 = rad_3(i) \cdot rad_2(i). \quad (4.6)$$

Like this in this way, function Walsh $wal_4(\theta)$ represents by yourself produced -

functions Rademacher $rad_2(\theta)$ And $rad_3(\theta)$.

On rice. 4.2 functions are presented Walsh For $N = 8$.



Rice. 4.2. Walsh system consisting from $N = 2^3 = 8$ functions

be complete and orthogonal, therefore it is suitable for decomposing signals of arbitrary type with finite interval of definition. Walsh functions are piecewise constant. Interval definitions functions Can consider -

vat consisting from $N = 2^m$ equal subintervals. On each from them functions Walsh accept values $+1$ or -1 . IN points discontinuity of the function is continuous on the right.

Functions Walsh orthogonal And standardized, because

$$\int_0^1 wal_n(\theta) wal_k(\theta) d\theta = \begin{cases} 1 & \text{at } n=k \\ 0 & \text{at } n \neq k \end{cases} \quad (4.7)$$

Average meaning functions Walsh For everyone $n \neq 0$ equals zero:

$$\int_0^1 wal_n(\theta) d\theta = 0, \quad n=1, 2, \dots, N-1. \quad (4.8)$$

Walsh functions have the *multiplicative property*, That There is work two functions Walsh equals new functions Walsh from the same system:

$$wal_n(\theta) \cdot wal_k(\theta) = wal_p(\theta). \quad (4.9)$$

Multiplication any functions Walsh the most on myself gives function With zero number $wal_0(\theta)$:

$$wal_n(\theta) \cdot wal_n(\theta) = wal_0(\theta). \quad (4.10)$$

4.3. Decomposition signals By functions Walsh

For decomposition signals, given on interval $[0, T)$, comfortable use functions Walsh, which after transformations their argument are being recorded view $wal_n(t/T)$.

Row Walsh one-dimensional signal $x(t)$, $t \in [0, T)$, will have view

$$x(t) = \sum_{n=0}^{\infty} c_n wal_n\left(\frac{t}{T}\right), \quad (4.11)$$

где

$$c_n = \frac{1}{T} \int_0^T x(t) wal_n\left(\frac{t}{T}\right) dt. \quad (4.12)$$

So How functions Walsh on intervals discreteness accept values +1 or -1, at calculation coefficients c_n Not required perform a multiplication operation. Therefore, Walsh spectral analysis requires less computational time than analysis using harmonic functions.

Truncated rows Walsh

$$x_N^*(t) = \sum_{n=0}^{N-1} c_n wal_n\left(\frac{t}{T}\right) \quad (4.13)$$

have uniform, mean-square and average convergence and can be used to approximate signals, described integrable functions. Charts functions $x^*(t)$ have a stepwise character.

^N

4.4. Methodological instructions

To make up program, suitable For analysis functions on intervals different duration, functions Walsh recommended build at first on interval $[0, 1)$ relative time. On this interval, taking into account the discontinuous nature of the functions, at least 1000 readings are placed. The Rademacher functions described by formula (4.1) can be form, having applied, conditional operator if. For example, For functions Rademacher $rad_3(t)$ we will receive

$$rad_3(t) = \text{if}(\sin(8 \cdot p \cdot t) > 0, 1, -1). \quad (4.14)$$

System functions Rademacher recommended write down V view century-torus functions, components which There is functions Rademacher $rad_i(t)$, $i = 1, \dots, m$.

The system of Walsh functions can be written as a vector function, the elements of which are Walsh functions and are expressed through functions Rademacher. For facilities All functions recommended write down relative time θ , changing V interval $[0, 1)$. To get functions Walsh For given interval $[0, T)$, V century-torus functions is being carried out substitution $t = t/T$. IN application P.6 Dan example formations like this way systems functions Haar.

4.4. Program work

4.4.1. The main thing exercise

1. Compose program modeling functions Walsh $wal_n(t)$, $t \in [0, T)$, $n = 0, 1, \dots, N-1$, For $N = 16$. Observe graphs functions Walsh And check, What They are formed correctly.

2. Make sure V volume, What functions Walsh satisfy orthogonality condition. To do this, calculate the values of the integrals

$$\frac{1}{T} \int_0^T wal_n\left(\frac{t}{T}\right) \cdot wal_m\left(\frac{t}{T}\right) \cdot dt$$

при нескольких произвольных значениях $n = m$ and $n \neq m$ ($n, m = 0, 1, \dots, N-1$).

3. Generate the given function $x(t)$ in the MathCAD environment,

$t \in [0, T)$. Plot its graph.

Note. Signal $x(t)$ is being asked teacher or defines from applications P.1 according to given option. The signal duration is chosen arbitrarily: $T = T_s \neq 1$ With.

4. Calculate values coefficients c_n , $n = 0, 1, \dots, N-1$, broke down - of the given functions $x(t)$ V basis functions Walsh.

5. Compose program calculations functions

$$x^*(t) = \sum_{n=0}^N c_n wal_n\left(\frac{t}{T}\right), \quad t \in [0, T).$$

6. Build spectral diagram And do conclusions O spectrum composition researched signal $x(t)$.

7. Build schedule mistakes approximations

$$\varepsilon_N(t) = x(t) - x_N^*(t).$$

Analyze characteristic signs mistakes approximations.

8. To educate approximating signal $x^*(t)$ by N way summation of first eight members rows ($N=8$). Build schedule mistakes approximations $\varepsilon_N(t) = x(t) - x_N^*(t)$. Do conclusions.

9. Calculate values average quadratic mistakes

$$\sigma^2 = \int_0^T [x(t) - x_N^*(t)]^2 dt$$

For $N=8$ And $N=16$. Do conclusions.

4.5. Control questions And tasks

1. What are the advantages of using Walsh functions over others?
2. Write it down at help functions Rademacher function Walsh $wal_{21}(\theta)$.
3. Show me, What functions Walsh orthogonal.
4. Show that the product of any two Walsh functions is also a Walsh function. What is this property called?
5. Educate function $wal_2(\theta) \cdot wal_6(\theta)$. Determine number functions Walsh, resulting V result this actions.
6. Study the graphs of the Walsh functions in Fig. 4.1 and find the pattern that governs the values of the functions on the last interval of constancy.
7. Name and explain the existing methods of ordering Walsh functions.
8. Specify the feature of Walsh functions that simplifies the calculation of the coefficients of the series.
9. How from systems discrete functions Walsh For $N=16$ image - vat similar system For $N=8$?

Practical lesson No. 5

DECOMPOSITION OF SIGNALS BY SYSTEM FUNCTIONS HAAR

5.1. Target work

, highly efficient compression algorithms for image processing are based primarily on wavelet analysis methods, among which the classical orthogonal Haar basis occupies a prominent place. Haar functions, like Walsh functions, belong to the class of piecewise constant functions. Their significant difference from Walsh functions is that they are localized to specific parts of the interval under study. Therefore, Haar functions, which allow one to evaluate the local properties of the signals under study, are often called Haar wavelets.

The aim of the work is to study the system of orthogonal Haar functions, decomposition of a signal of a given shape and study the influence of the number of terms of the series on the approximation error.

5.2. Main concepts And settlement formulas

5.2.1. Functions Haar

was first constructed and studied by V 1909 G. A. Haar. System Haar consists of from piecewise constants functions, given on interval $[0, 1)$. For functions Haar along With Single (ordinal) numbering is also widely used, as is double numbering. Double numbering corresponds to the group principle of forming Haar functions.

First function systems Haar constant:

$$\chi_0(\theta) \equiv 1, \quad \theta \in [0, 1). \quad (5.1)$$

Rest functions systems Haar comfortable build By groups:

group number m contains 2^m functions $\chi_{mk}(\theta)$, $m = 1, 2, \dots$,
 $k = 0, 1, \dots, 2^m - 1$.

IN first group ($m = 0$) enters one function:

$$\chi_1(\theta) = \begin{cases} 1, & \text{if } 0 \leq \theta < 0.5 \\ 0, & \text{if } 0.5 \leq \theta < 1 \end{cases} \quad (\theta) =$$

$$\theta \in [0, 1);$$

1

2

$$\begin{cases} 1, & \theta \in [0, \frac{1}{2}), \\ -1, & \theta \in [\frac{1}{2}, 1). \end{cases}$$

The second group ($m=1, k=0.1$) consists of two functions:

$\chi_2(\theta) = \chi_{10}(\theta)$ And $\chi_3(\theta) = \chi_{11}(\theta)$; third group ($m=2, k=0.1, 2, 3$) - from four functions: $\chi_4(\theta) = \chi_{20}(\theta)$, $\chi_5(\theta) = \chi_{21}(\theta)$,

$$\chi_6(\theta) = \chi_{22}(\theta),$$

$$\chi_7(\theta) = \chi_{23}(\theta) \text{ And etc.}$$

For formations functions Haar is used next formula :

In Fig. 5.1 shows the first eight functions of the Haar system. It is easy to see that the first two Haar functions are the same as the corresponding Walsh functions, ordered according to Walsh or Paley . The remaining Haar functions have a local character of the domain of definition .

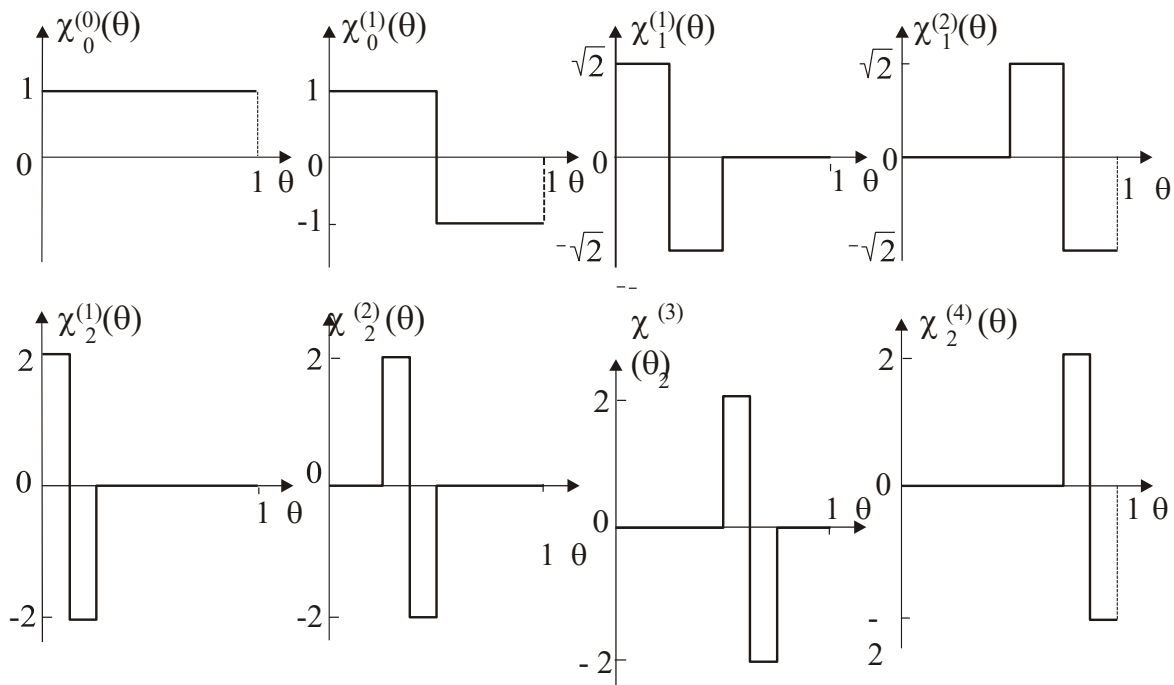


Fig.5.1. Charts first eight functions Haar

Nowadays, Haar functions are also called Haar wavelets, since they can

be described using the formal rules adopted V theories wavelet analysis.
Scaling function And "maternal

The Haar wavelet functions coincide with the first and second functions of the system under consideration, that is:

$$\varphi(\theta) = \chi_0(\theta) \equiv 1, \quad \theta \in [0, 1); \quad (5.4)$$

$$\Psi(\theta) = \varphi(2\theta) - \varphi(2\theta - 1) = \chi_1(\theta) = \begin{cases} 1, & \theta \in [0, \frac{1}{2}); \\ -1, & \theta \in [\frac{1}{2}, 1). \end{cases} \quad (5.5)$$

The functions of the Haar system are determined according to the wavelet theory by scaling transformations and translations of the “mother wavelet”:

$$\chi_{mk}(\theta) = \sqrt{2^m} \psi(2^{-m}\theta - k); \quad m = 0, 1, 2, \dots; \quad k = 0, 1, \dots, 2^m - 1. \quad (5.6)$$

At use single numbering, How easily show, But-measures functions Haara, starting With $n = 1$, is determined By values m And k With help formulas

$$n = 2^m + k; \quad m = 0, 1, 2, \dots; \quad k = 0, 1, \dots, 2^m - 1. \quad (5.7)$$

Sometimes Haar functions are defined differently. On intervals where the functions differ from zero, their values are taken to be equal to +1. or -1. The Haar functions defined in this way are orthogonal but not normalized. Such unnormalized Haar functions are convenient to use in the analysis and synthesis of logical functions.

5.2.2. Decomposition continuous signals V basis Haar

The system of orthogonal Haar functions can be used as a basis for expanding a continuous signal defined on the interval $[0, T]$ into a uniformly convergent Haar series.).

At use functions Haar V quality basic For approximations signal $x(t)$ on segment $[0, T)$ dimensionless argument θ necessary replace on αt , Where coefficient $\alpha = 1/T$ asks necessary - my temporal scale functions And has dimension time V minus of the first degree.

Haar series one-dimensional signal $x(t), t \in [0, T)$ will have view

have uniform, mean-square, and average convergence. They can be used to approximate signals described by integrable functions.

The root mean square error of approximation for a finite number of orthogonal components of the Haar series is calculated using the formula

$$\int_0^T \left| x(t) - \sum_{n=1}^{N-1} \chi_n(t) \right|^2 dt$$

$$\sigma^2 = - \int_0^T \left| x(t) - \sum_{n=0}^{\infty} c_n \chi_n \left(\frac{t}{T} \right) \right|^2 dt. \quad (5.11)$$

At use double numbering row Haar is written as follows:

$$x(t) = c_0 + \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} c_{mk} \chi_{mk} \left(\frac{t}{T} \right). \quad (5.12)$$

A distinctive feature of Haar functions is their relative ease of generation in radio devices. The basic system of Haar functions is useful for analyzing and synthesizing finite-duration pulse signals .

The use of Haar functions is most effective for analyzing signals with strongly pronounced local features in the form of short- term bursts and oscillations. This is explained by the fact that the approximation of these bursts and oscillations is carried out by a limited number of components of the series located in the corresponding part of the interval . $[0, T)$.

5.3. Methodological instructions

Program work provides for decomposition V row Haar whitefish-Nala, described on given interval $[0, T)$ function $x(t)$. To make up program, suitable For analysis functions on intervals of different duration, functions Haar recommended form at first on interval $[0, 1)$ relative time. At compilation programs modeling functions Haar offered use "mother wavelet" described by function (5.5). The "mother wavelet" on a relative time scale can be obtained , in particular, using the conditional operator if :

$$h(t) := \text{if}(t \leq 0, 0, 1) - 2 \text{if}(t \leq 0.5, 0, 1) + \text{if}(t \leq 1, 0, 1) .$$

One of the possible ways to obtain a system of Haar functions is to form a vector function, each component of which is written according to formula (5.6) on a relative scale time. And With compliance single numbering. To get functions Haar For given interval $[0, T)$, V vector-functions is being carried out indicated higher substitution $t \neq t T$.

Example formations vector functions Haar given V Appendix P.6 .

5.4. Program work

5.4.1. The main thing exercise

1. Compose program modeling functions Haar $\chi_n(t)$, $t \in [0, T)$, $n = 0, 1, \dots, N-1$, For $N = 16$. Observe graphs functions Haar And check their correctness.

2. Make sure V volume, What functions Haar satisfy condition orthogonality . To do this , calculate the values of the integrals for several arbitrary values $n = m$ and $n \neq m$ ($n, m = 0, 1, \dots, N-1$).

3. Generate the given function $x(t)$ in the MathCAD environment ,
 $t \in [0, T)$. Build her schedule.

Note. Signal $x(t)$ is being asked teacher or define-etsya from applications P.1 according to given option. The signal duration is chosen arbitrarily: $T = T_s \neq 1$ With .

4. Calculate values coefficients c_n , $n = 0, 1, \dots, N-1$, broke down - zheniya given functions $x(t)$ V basis Haar functions .

5. Compose program calculations functions

6. sharpen spectral diagram And do conclusions O spec-minesweeper composition researched signal $x(t)$.

7. Build schedule mistakes approximations

$$\varepsilon_N(t) = x(t) - x_N^*(t) .$$

Analyze characteristic signs mistakes approximations.

8. To educate approximating signal $x^*(t)$ by way summirova - nia first eight members rows ($N=8$). Build schedule mistakes approximations $\varepsilon_N(t) = x(t) - x_N^*(t)$. Do conclusions.

9. Calculate values average quadratic mistakes

$$\sigma^2 = \int_0^T [x(t) - x_N^*(t)]^2 dt$$

For $N=8$ And $N=16$. Draw conclusions.

5.5. Control questions And tasks

1. Give it to me definition functions Haar.

2. Depict schedule functions $x(\theta) = \chi_3(\theta) + \chi_7(\theta)$.

3. Graph the function $x(t) = 2 \begin{pmatrix} t \\ T \end{pmatrix} + \chi_6 \begin{pmatrix} t \\ T \end{pmatrix}$, если

$T=0,1$ с.

4. if it uses four groups ?

5. How many functions Haar enters V group $m=6$?

6. To which group m does the Haar function belong $\chi_{15}(\theta)$? Image - and its graph?

7. Show me, What functions Haar orthogonal.

8. Find energy functions Haar χ_3 at $T=2$ With.

Practical lesson No. 6

STUDY ANALOGUE LOW-PASS AND HIGH-PASS FILTERS

6.1. Target work

The solution to the problem of approximating the characteristics of ideal low-pass and high-pass filters (LPF and HPF) using a stable and physically feasible transfer function led to the creation of an entire family of standard filters. The most well-known of these, due to their simplicity, are the Butterworth and Chebyshev filters.

The aim of the work is to study the methods of calculating low-pass filters. And HPF Butterworth and Chebyshev according to the given requirements for the amplitude - frequency response (AFR) and construction of the characteristics of the calculated filters .

6.2. Main concepts And settlement formulas

6.2.1. Standardized analog filters lower frequencies

The normalized transfer functions of the Butterworth and Chebyshev filters (see Appendix P.2) can be represented as

$$H_{on}(s) = \frac{b_0}{A(s)} = \frac{b_0}{s^n + a_{n-1} \cdot s^{n-1} + \dots + a_1 \cdot s + a_0} \quad (6.1)$$

The properties of these filters are described in detail in the literature. There are also other filters that have transfer functions of the form (6.1) and combine the properties of the Butterworth and Chebyshev filters. The Legendre filter , for example, has a frequency response slope in the attenuation band that is as steep as the Chebyshev filter, and monotonous frequency response V bandwidth. In The Butterworth -Thomson filter combines the maximally flat frequency response of the Butterworth filter and the linear phase response of the Bessel filter.

Typically, a filter is implemented as a series connection of first- and second-order links. Therefore, the denominator of the transfer function (6.1) is represented as a product of factors no higher than the second order. Then

$$A(s) = \begin{cases} \prod_{i=1}^{n/2} (s^2 + \alpha_{1i}s + \alpha_{2i}) & \text{at even } n, \\ \prod_{i=1}^{(n-1)/2} (s - \alpha_i) & \text{at odd } n \end{cases} \quad (6.2)$$

$$\prod_{i=1}^n (s^2 + \alpha_{1i}s + \alpha_{2i}) \quad \text{at odd } n.$$

6.2.2. Filter Butterworth

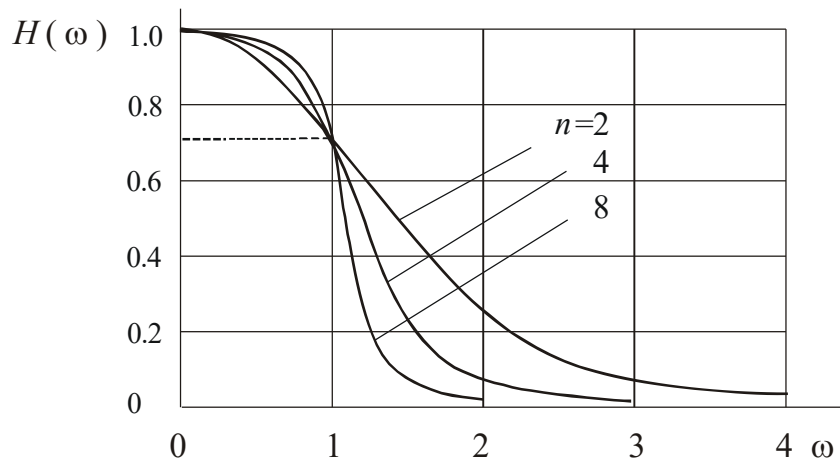
frequency response filter Butterworth is described expression

$$H(\omega) = \frac{1}{\sqrt{1 + (\omega/\omega_c)^{2n}}}, \quad (6.3)$$

Where ω_s - border frequency, n - filter order .

Butterworth filters have the flattest frequency response in the passband. And monotonous characteristics V strip delays.

frequency response filters Butterworth For $n = 2, 4$ And 8 shown on rice. 6.1.



Rice. 6.1. Frequency response normalized filters Butterworth

As the order of n increases The Butterworth filter 's passband gain increasingly approaches unity, transitional region All V greater degrees narrows, A V polo- se delays function transmissions All closer And closer fits To Well- love . At $n \rightarrow \infty$ frequency response filter Butterworth is approaching To ideal.

Filter Butterworth has next properties.

1. At any n the following relations are valid :

$$H(0) = 1; \quad H(\omega_c) = 1/\sqrt{2}; \quad \lim_{\omega \rightarrow \infty} H(\omega) = 0. \quad (6.4)$$

2. Function $H(\omega)$ filters Butterworth monotonously is decreasing at $\omega \geq 0$. Hence, frequency response has maximum meaning at $\omega = 0$.

3. All derivatives functions (6.3) By frequency ω from first to $(n-1)$ -y inclusive at $\omega = 0$ equal zero. Precisely because this reason - Not filter Butterworth they call filter With maximum flat Frequency response.

4. Steepness frequency response filter Butterworth n - th order on at high frequencies is $20n$ dB/dec.

Like this in this way, n And ω_s are those parameters, choice koto- loose allows satisfy given kit requirements To filter in the passband and stopband.

Transfer function filter Butterworth Not has zeros, A her n poles are located V left half-planes on circles With radio - catfish, equal ω_s .

6.2.3. Filter Chebysheva first genus

Amplitude-frequency characteristic low-pass filter Chebysheva the first kind is determined by the expression:

$$H(\omega) = \frac{1}{\sqrt{1 + \varepsilon^2 V_n^2(\omega/\omega_c)}}, \quad (6.5)$$

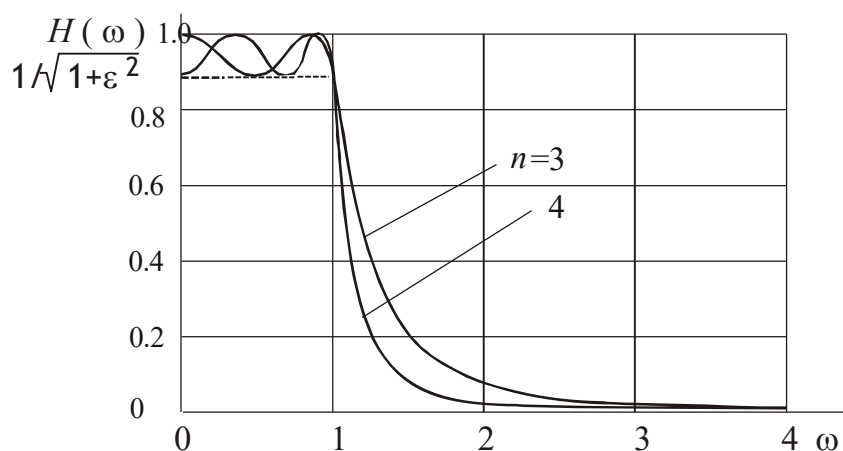
where V_n – Chebyshev polynomial of order n , which can be formed using a recurrence formula

$$V_{n+1}(x) - 2xV_n(x) + V_{n-1}(x) = 0. \quad (6.6)$$

Here the first two polynomials are taken to be equal: $V_0(x) = 1$; $V_1(x) = x$ On . rice. 6.2 shown frequency response filter Chebysheva first genus. The frequency response has equal in size pulsations V strip transmission And monotonous Ha- characteristics V strip detentions. Scope pulsations frequency response equal

$$\delta = 1 - 1/\sqrt{1 + \varepsilon^2}. \quad (6.7)$$

Like this in this way, ε represents by yourself free parameter, koto- rry sets size unevenness transmissions V strip transmission . How less meaning ε , those less width stripes, V ko- second fluctuates frequency response V strip transmission.



Rice. 6.2. frequency response standardized Chebyshev filters of the first type (ε)

=0.765)

Filter Chebysheva has next properties:

1. frequency response satisfies conditions:
2. On interval $[0, \omega_s]$ values of functions $H(\omega)$ are lying down V within from $\frac{1}{\sqrt{1+\varepsilon^2}}$ to 1. IN general difficulties, on this interval there is n critical points, V which function $H(\omega)$ reaches minimal-go values $\frac{1}{\sqrt{1+\varepsilon^2}}$, or maximum values, equal 1.

3. At $\omega \geq \omega_s$ function $H(\omega)$ monotonously is decreasing And strives To Well-love . Steepness decline on high frequencies makes up $20 n$ dB/dec

Choosing values of n and ε , it is possible to satisfy a given set of filter requirements in passband and stopband . Transfer function filter Chebysheva per-

vogo genus Not has zeros. All poles are located V left half-planes on an ellipse, the major axis of which is located on the ordinate axis, and the minor axis on the abscissa axis.

6.2.4. Denormalization And transformation filters

The design of low-pass and high-pass filters begins with the determination of the transfer function of the *normalized* low-pass filter. Under the normalized filter, this implied filter With frequency cut stripes transmission

$\omega_c = 1$. Transfer functions filters With given frequencies cut obtained using *denormalization operations* (scaling by frequency) And *transformations* (filter type transformation).

To obtain the transfer function of the low-pass filter with the required cutoff frequency, the *denormalization operation is used* (*scaling by frequency*). IN transmission functions standardized low-pass filter opera- Thor s is replaced by operator s/ω_c , That There is sought after transmission the LPF function will be equal to

$$H(s) = H_H(s/\omega_c). \quad (6.10)$$

In order to obtain the transfer function of the high-pass filter with the required cutoff frequency, denormalization operations are carried out simultaneously And transformations. Specified operations are being carried out by way for-

exchanges operator s V transmission functions standardized low-pass filter on operator ω_c/s . Transfer function HPF will be determined So:

$$H(s) = H_n(s/\omega_s). \quad (6.11)$$

6.3. Methodological instructions

The paper examines the frequency characteristics of low-pass and high-pass filters. The frequency characteristics are calculated directly from the frequency transfer function.

$$H(j\omega) = H(s)_{s=j\omega}.$$

Frequency ω at construction frequency characteristics It is recommended to change in the range from 0 to $3\omega_s$.

For calculation poles filter are used existing V system MathCAD functions **root** (**f** (**x**), **x**) or **polyroots** (**v**). Example definitions roots of a polynomial using the function **polyroots** (**v**) given in the appendix P.6.

Pulse transitional function is determined How the opposite pre-education Laplace transmission functions $H(s) = B(s)/A(s)$:

$$h(t) = L^{-1}\{H(s)\} = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} H(s)e^{st} ds. \quad (6.12)$$

Taking into account that the poles of the filters studied in this work are simple and non-zero, the impulse response function can be defined in the following form:

$$h(t) = \sum_{i=1}^n C_i \cdot e^{s_i \cdot t}, \quad (6.13)$$

Where s_i - poles filter; C_i - constant, calculated By formula

$$C_i = \frac{B(s_i)}{A'(s_i)}. \quad (6.14)$$

6.4. Work program

6.4.1. The main thing exercise

1. Calculate coefficients transmission functions low-pass filter And HPF 2nd And 3rd order With given type And meaning frequencies cut ω_s .

Note: The types of filters under study and the cutoff frequency value are suggested by the teacher or, by agreement, determined by the student

according to the option from Table 6.1.

Table 6.1

Type And filter parameters		Numbers options							
		1	2	3	4	5	6	7	8
Type FNF	–	Bath	Cheb1	Bath	Cheb2	Bath	Cheb3	Bath	Cheb4
Type HPF	–	Cheb1	Bath	Cheb2	Bath	Cheb3	Bath	Cheb4	Bath
ω with	work/s	150	135	120	105	90	75	60	45

2. Compose programs For calculation frequency response $H(\omega)$ And phase response $\varphi(\omega)$ filters.

3. Build frequency response $H(\omega)$ And phase response $\varphi(\omega)$ filters. Compare characteristics of the same type filters various orders And do conclusions.

3. Calculate the poles of the filters under study and show the location of the poles on the complex plane. Compare the location of the poles of similar filters of different orders and draw conclusions.

4. Calculate And build pulsed transitional functions $w(t)$ filters. Compare the characteristics of similar filters and draw conclusions.

6.5. Control questions And tasks

1. Give it to me definition amplitude-frequency characteristics.
2. Explain the concepts of passband and stopband of a filter.
3. Give it to me concepts filters lower And top frequencies.
4. Find meaning ε , at which magnitude The transmission unevenness in the passband of the Chebyshev filter is 0.1.
4. Explain meaning operations denormation .
5. Explain meaning operations transformations.
6. Explain how to transform the transfer function of a low-pass filter into the transfer function of a high-pass filter.
7. How do the frequency responses of Butterworth and Chebyshev filters differ in the passband ?

Practical lesson No. 7

PERFORMANCE SIGNALS AT HELP FROM A NUMBER OF KOTELNIKOV'S

7.1. Target work

The first stage of digital signal processing is sampling. As a result of sampling, the continuous signal is replaced by samples taken at certain, usually equal, time intervals. In this case, the choice of the sampling interval is very important. With a large sampling interval, rapid changes in the continuous signal may remain unnoticed. Choosing a small sampling interval leads to an increase in the number of samples and thereby to an increase in processing time. The problem of choosing the sampling interval most just is being decided at help theorems, proven by V. A. Kotelnikov in 1933. This theorem establishes the conditions for the exact reconstruction of instantaneous values of a signal from its readings taken at equal intervals of time.

The aim of the work is to determine the sampling period based on the spectral characteristics of a given signal and to estimate the error in signal reconstruction using the Kotelnikov series.

7.2. Main concepts And settlement formulas

Signal $x(t)$ and spectral characteristics $X(j\omega)$ are mutually one-to-one related by the direct and inverse Fourier transforms:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt; \quad (7.1)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega. \quad (7.2)$$

Let the spectral characteristic $X(j\omega)$ is equal to zero at $\omega < -\omega_{in}$ And $\omega > \omega_{in}$. In this case we can transform V periodically - I'll With period $W = 2\omega_{in}$ And introduce V in the form of a complex rows Ugh-rye. Replacing V formula complex rows Fourier variable t on ω And period T on $W = 2\omega_{in}$ we will receive

$$X(j\omega) = \frac{1}{2} \sum_{n=-\infty}^{\infty} A_n e^{jn \frac{2\pi}{W} \omega},$$

(7.3)

Where coefficients decomposition

At this spectral characteristic (7.3) will accept view

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x(nT) e^{jnT\omega} = T \sum_{n=-\infty}^{\infty} x(nT) e^{-jnT\omega}. \quad (7.7)$$

Substituting (7.7) into (7.2), changing the order of integration and summation and calculating the integrals, we find

On basis stated IN. A. Kotelnikov was The following was formulated

Theorem. I love it function $x(t)$, containing harmonic components With frequencies from 0 to ω_{in} , Can introduce V view rows (7.8) And, vice versa, any function, presented formula (7.8), so-holds only harmonic components With frequencies from 0 to ω_{in} .

Received formula (7.8) Can consider How decomposition signal $x(t)$ By functions

$$\varphi_{nT}(t) = \frac{\sin \omega_{in}(t - nT)}{\omega_{in}}, \quad (7.9)$$

and V quality coefficients rows perform values signal

$x(t)$ V discrete moments time $t_n = nT$, $n \in (-\infty, \infty)$.

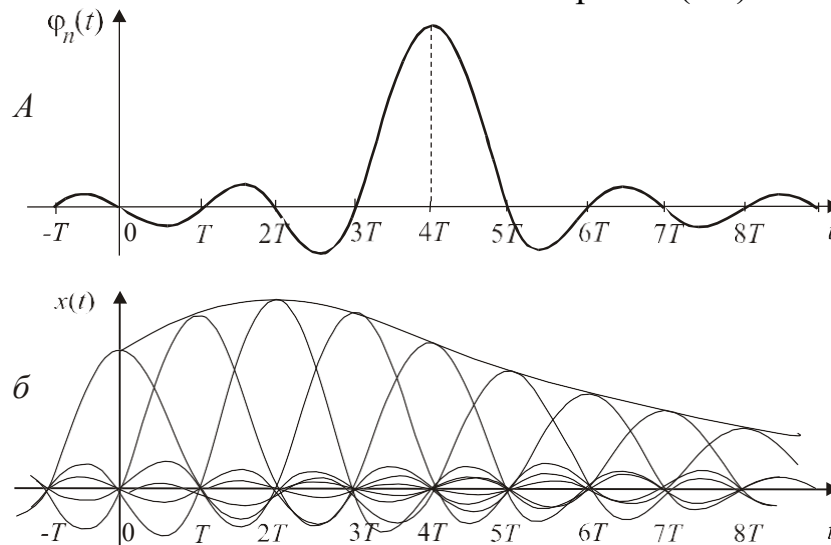
Function $\varphi_n(t)$, called reference function, displays so-
 fight fluctuations With maximum meaning at $t_n = nT$ (on rice. 7.1, a
 presented schedule reference functions For $n=4$) . IN other discrete -
 nye moments time function equal zero. Easily check, What the counting functions
 are orthogonal on the time interval, that is,

$$\int_{-\infty}^{\infty} \varphi_n(t) \varphi_m(t) dt = \begin{cases} d_n & \text{при } n=m, \\ 0 & \text{при } n \neq m, \end{cases} \quad (7.10)$$

Wh
 ere

$$d_n = \int_{-\infty}^{\infty} \left[\frac{\sin \omega (t-nT)}{\omega (t-nT)} \right]^2 dt = \frac{\pi}{\omega_{in}} = T. \quad (7.11)$$

Performance functions $x(t)$ near (7.8) shown on rice. 7.1,b.
 IN each point $t = nT$ only one member rows, standing V right parts
 expressions (7.8), excellent from zero And this member equal $x(nT)$. Sledova -
 tentially , V points $t = nT$ justice formulas (7.8) is obvious. IN pro-
 in between between specified dots accurate meaning functions $x(t)$
 provided summation infinite numbers functions species (7.9).



Rice. 7.1. Performance signal V view rows Kotelnikova:
 A - reference function; b - components rows

Like this in this way, function $x(t)$ With limited spectrum, With one
 sides, Maybe be fully given multitude her instant values, taken through
 equal intervals time T . WITH another side -
 we , If there are numeric values functions $x(nT)$ For everyone n , That she
 Maybe be fully restored By formula (7.8).

In the practical use of Kotelnikov's theorem for reconstructing a continuous signal from its samples, the following must be taken into account. Signals with a limited spectrum, for which the theorem is valid, are infinite in time. Real signals are limited in time by the interval $[0, T_s]$ and therefore have an unlimited time spectrum. However, it is always possible to identify an interval frequencies $[0, \omega_{in}]$, V which concluded main Part energy signal, A on share components spectrum With frequency $\omega > \omega_{in}$ have to a small part of the signal energy.

Signal, limited By time, approximately is described series (7.8), consisting of a finite number of members:

At summation members rows (7.12) signal $x(t)$ is being reproduced exactly only V points countdown $t_n = nT$. IN intervals between countdown- an approximation error occurs, the magnitude of which depends on the discarded part of the signal spectrum. To reduce the error, the sampling interval T It is recommended to take 2–5 times less than the value determined by formula (7.6). The Kotelnikov series is rarely used for signal reconstruction. It has more theoretical significance, allowing one to draw useful conclusions. In practice, other methods are more often used to reconstruct signals from discrete readings, for example, linear and quadratic interpolation methods.

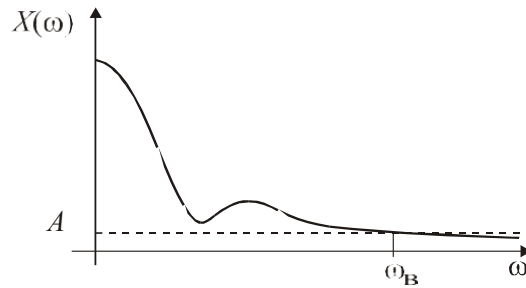
7.3. Methodological instructions

IN work is being investigated signal $x(t)$, representing by yourself alone - pulse of a given shape (see Appendix P.1). The operating program provides for the sampling of a given signal, its reconstruction from discrete samples, and the evaluation of the reconstruction error.

Period discretization T located By formula (7.6). For defined - leniya border frequencies ω_{in} , which is used V formula (7.6), is being built amplitude spectral characteristic $X(j\omega) = |X(j\omega)|$ signal. By this characteristics is determined border interval ω_{in} , V which concluded main Part energy signal (rice. 7.2). Meaning - nie ω_{in} is located from conditions, what if $\omega > \omega_{in}$ is being carried out inequality

$$X(\omega) \leq A,$$

Where A - given value.



Rice. 7.2. Definition borders frequency interval

At compilation programs recovery signal By formula (7.12) necessary accept in attention, What incoming V her counting functions (7.9) have uncertainty at $t = nT$. To eliminate this one Not-certainty, it is recommended to use the conditional operator in the program, which is part of the MatchCAD system. For example, the program for calculating the function

$$y(x) = \sin x / x \quad (7.13)$$

With help conditional operator Maybe be recorded in the form

$$y(x) := \text{if}(x \approx 0, 1, \sin x / x). \quad (7.14)$$

How it was noted higher, row (7.8) exactly reproduces signal $x(t)$ V points countdown $t_n = nT$. Hence, V these points error approx - the simulation is equal to zero, the error maxima should be expect approximately in the middle of the intervals between adjacent reference points. Therefore, to quickly estimate the approximation error, one can calculate the value - nia mistakes $\varepsilon(nT + 0.5T)$.

7.4. Work program

7.4.1. The main thing exercise

1. Find spectral characteristics reference functions

$$\varphi_n(t) = \frac{\sin \omega_{\text{in}}(t - nT)}{\omega_{\text{in}}(t - nT)}.$$

Build her For arbitrary values n And $T = \pi / \omega_{\text{in}} = 1$.

2. Form V environment MatchCAD mathematical model signal $x(t)$, given teacher (options signals given V application P.1). Build schedule functions $x(t)$.

3. Compose program calculation spectral characteristics

$$X(j\omega) = \int_0^{\infty} x(t)e^{-j\omega t} dt$$

given signal $x(t)$. Build amplitude spectral characteristics $X(\omega) = |X(j\omega)|$.

4. By amplitude spectral characteristics $X(\omega)$ define meaning frequencies from $X(\omega) \leq 0.05 \max_{\omega} X(\omega)$ at $\omega \geq \omega_{in}$.
 ω_{in} conditions, What ω

5. Define interval discretization Kotelnikova $T = \pi / \omega_{in}$
 And form discrete subsequence

$$x(n) = x(nT), n = 0, 1, \dots, N = T_c / T,$$

Where T_s - interval definitions of the given signal $x(t)$.

6. Compose program calculation signal $x^*(t_N)$, recoverable from discrete sequences $x(nT), n = 0, 1, \dots, N$, at help row -
 Yes Kotelnikova. Build on one drawing graphs functions $x(t)$
 And $x^*(t)$.

7. Form signal errors $\varepsilon(t) = x(t) - x^*(t)$. Define maximum absolute mistake

$$\varepsilon_{\max} = \max_t |\varepsilon(t)|.$$

8. Repeat points 4–6 programs, having reduced period discretization T 2–5 times.

7.5. Control questions And tasks

1. How is being carried out sampling continuous signal By time?
2. Show me, What counting functions $\varphi_n(t)$ orthogonal on in-time interval $(-\infty, \infty)$.
3. How are changing counting functions $\varphi_n(t)$ at decrease (increase) period discretization?
4. How looks like spectral characteristic reference functions?
5. What understand under number degrees freedom of signal $x(t)$?
6. What methods, besides the Kotelnikov series, are used to reconstruct a signal from its readings at discrete moments in time?

Practical lesson No. 8 DISCRETE INTEGRATION AND DIFFERENTIATION

8.1. Target work

In signal processing tasks, the need often arises to transform signals by integrating and differentiating them. For example, in control systems, derivative and integral effects are introduced to improve the control process. In medical information systems, obtaining diagnostic indicators is based on the integration or differentiation of signals. Such systems require digital integration and differentiation algorithms implemented in real time.

The goal work is studying some algorithms digital integration And differentiation signals, presented V in the form of finite discrete sequences.

8.2. Main concepts And settlement formulas

8.2.1. Discrete integration

Integration continuous signal is described equation

$$y(t) = \int_0^t x(\tau) d\tau. \quad (8.1)$$

for the approximate implementation of integration in discrete form. We will limit the class of discrete integration algorithms under consideration to algorithms that can be described by a difference equation.

$$y(n) = y(n-1) + \delta y[x(n), x(n-1), T], \quad (8.2)$$

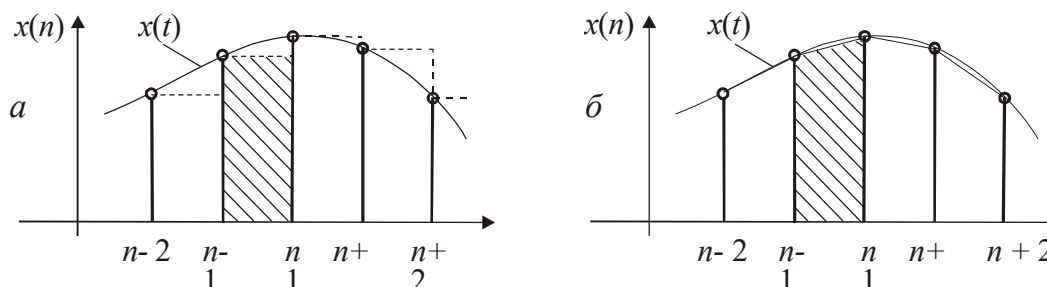
Where $y(n)$ - day off subsequence, representing by yourself assessment - ku integral, $\delta y[x(n), x(n-1), T]$ - magnitude increments on next sampling interval, depending on the integration method used.

Integration by the rectangle method. The increment value V equation (8.1) located How square rectangle (rice. 8.1, a). In this case, the difference equation of the discrete integrator takes the form

$$y(n) = y(n-1) + T \cdot x(n-1). \quad (8.3)$$

Having been subjected to equation z -transformation, we will receive transfer function of a discrete integrator

$$H_p(z) = \frac{Y(z)}{X(z)} = \frac{T \cdot z^{-1}}{1 - z^{-1}} \quad (8.4)$$



Rice. 8.1. Geometric performance integration:
A - By rectangle method, b - By method trapezoids

Trapezoidal integration . Increment of the integral in the equation (8.1) numerically equals area trapezoids, shown in Fig. 8.1, b). A discrete integrator implementing integration using the trapezoidal method is described by a difference equation

$$y(n) = y(n-1) + \frac{T}{2} \cdot [x(n-1) + x(n)]. \quad (8.5)$$

Having been subjected to This equation z -transformation, we'll find transfer function of a discrete integrator

$$H_T(z) = \frac{Y(z)}{X(z)} = \frac{T}{2} \cdot \frac{1 + z^{-1}}{1 - z^{-1}}. \quad (8.6)$$

Integration by a combined method. The frequency responses of discrete integrators implementing integration by the rectangular and trapezoidal methods are located above and below the frequency response of an ideal integrator, respectively. Therefore, it was proposed¹ approximate the frequency response of an ideal integrator by a weighted combination of the specified discrete integrators:

$$H_k(\omega) = \frac{3}{4} \cdot H_n(\omega) + \frac{1}{4} \cdot H_t(\omega). \quad (8.7)$$

After substitutions (8.4) And (8.6) we obtain

$$H_{To}(z) = \frac{T}{8} \cdot \frac{1 + 7z^{-1}}{1 - z^{-1}}. \quad (8.8)$$

¹ Al-Alaoui M. A. Novel digital integrator and differentiator // Electronics Letters. – 1993. – Vol. 29. № 4. – P. 376–378.

Integration By method parabolic approximations.

Idea method consists of V volume, What integrable function $x(t)$ on inter-
Vale $[(n-2)T, nT]$ is approximated parabola By existing three
values $x(n-2)$, $x(n-1)$ And $x(n)$ signal. Increment V formula
(8.2) located integration approximating functions on inter- interval $[(n-1)T, nT]$.

Difference equation such discrete integrator has the form

$$y(n) = y(n-1) + T \cdot \left[\frac{x(n) + x(n-1) - x(n-2)}{12} \right]. \quad (8.9)$$

Transfer function integrator equal

$$H_{\text{couple}}(f) = \frac{T}{12} \cdot \frac{5 + 8of^{-1} - from^{-2}}{1 - from^{-1}}. \quad (8.10)$$

8.2.2. Discrete differentiation

The ideal differentiation of a continuous signal is determined by the
expression

$$y(t) = \frac{dx(t)}{dt}. \quad (8.11)$$

How it is known, operators differentiation s And shift z are related by the ratio

$$s = \frac{1}{T} \ln z. \quad (8.12)$$

Function (8.12) can be expanded in a series in three ways. The first
terms of these expansions could be considered as variants of the description
of differentiation algorithms. However, one of these expansions leads to an
unstable differentiation algorithm, and the other to a differentiation algorithm
that cannot be implemented in real-time systems. The third expansion method
leads to the following two algorithms.

Differentiation by the simple difference method. We write function
(8.12) as a series:

Having held back V (8.13) first term, we will receive transmission
digital differentiator function

From here we'll find difference equation

$$y(n) \equiv \frac{1}{T} [x(n) - x(n-1)]. \quad (8.15)$$

Differentiating non-recursive filter. If in (8.13) take into account two first member rows, That after transformations we obtain

$$\frac{H_{\text{days}}(z)}{z} \equiv \frac{Y(z)}{2T} (3 - 4z^{-1} + z^{-2}). \quad (8.16)$$

$$= X(z)$$

Transfer functions (8.14) corresponds difference equation

$$y(n) = \frac{1}{2T} [3 \cdot x(n) - 4 \cdot x(n-1) + x(n-2)]. \quad (8.17)$$

Differentiating recursive filter. Considering differentiation How action, the opposite integration, let's consider filter, transfer function whom received inversion (8.8). This

filter unstable, So How has pole $z_1 = 7$. To get stable-

filter, it is proposed to ^{map} the unstable pole to the region inside the unit circle and change the transfer coefficient. As a result, the transfer function of the discrete differentiator takes the form

$$H_{\text{etc.}}(z) = \frac{8}{7T} \cdot \frac{1 - z^{-1}}{1 + \frac{1}{7}z^{-1}}. \quad (8.18)$$

From here we will receive difference equation

$$y(n) = \frac{1}{8} y(n-1) + \frac{1}{7T} [x(n) - x(n-1)]. \quad (8.19)$$

8.3. Methodological instructions

This paper examines the discrete integration and differentiation algorithms described above to determine whether their characteristics match those of ideal algorithms. To this end, two types of research are proposed:

- phase response of discrete integrators and differentiators and comparison of these characteristics with the corresponding characteristics of an ideal integrator and differentiator;
- calculation of the response of discrete integrators and differentiators to the input test sequence and comparison with the response of an ideal integrator and differentiator.

² There same.

ACh $H(\omega) = |H(e^{j\omega T})|$ and FCH $\varphi(\omega) = \arg(H(e^{j\omega T}))$ recommended build V interval $[0, \pi]$ standardized frequencies $\tilde{\omega} = \omega T$.

An ideal integrator has a frequency response and phase response determined by the expressions :

$$H_{\text{and}}(\omega) = \left| \frac{1}{j\omega} \right| = \frac{1}{\omega} \quad \varphi(\omega) = -\frac{\pi}{2} \quad (8.20)$$

frequency response And phase response ideal differentiator are described formula-

we:

$$H_d(\omega) = |j\omega| = \omega \quad \varphi(\omega) = \frac{\pi}{2} \quad (8.21)$$

A harmonic discrete sequence is used as a test sequence for studying discrete integrators and differentiators in the time domain .

$$x(n) = \sin\left(\frac{2\pi}{M}n\right) \quad (8.22)$$

Discrete subsequence $y(n)$ on exit discrete inte - output of an integrator (or differentiator) is calculated directly using the difference equation. The calculated sequence is compared with the discrete sequence that could be obtained by discretizing the output signal of an ideal integrator (or differentiator). It is easy to show that these sequences are described by the expressions :

When comparing the corresponding sequences, the presence of a transient process should be taken into account.

8.4. Work program

8.4.1. The main thing exercise

1. Compose programs calculation frequency response $H(\omega)$ And phase response $\varphi(\omega)$ discrete -

integrators implementing integration using the rectangular and trapezoidal methods. Construct the frequency response and phase response of the integrators and compare them with the same characteristics of an ideal integrator, plotted on the same graphs.

2. Compose programs calculation discrete sequences $y(n)$ on exit discrete integrators, implementing integration By methods rectangles And trapezoids. Build graphs $y(n)$ And compare their With graphs $y_{\text{and}}(n)$ ideal integrator.

3. Compose programs calculation frequency response $H(\omega)$ And phase response $\varphi(\omega)$ discrete - differentiator implementing the simple difference method. Construct the frequency response and phase response of the differentiator and compare them with the same characteristics of an ideal differentiator, plotted on the same graphs.

4. Compose programs calculation discrete sequences $y(n)$ on exit discrete differentiator, implementing method simple differences. Build graphs $y(n)$ And compare their With graphs $y_{\text{and}}(n)$ ideal differentiator.

5. By similar program research:

- discrete integrators, described transmission functions (8.8) and (8.10);
- discrete differentiators described by transfer functions (8.16) and (8.18).

8.5. Control questions And tasks

1. Dan harmonic signal $x(t) = 10 \cdot \sin(0.5 \cdot t)$. Implemented sampling signal With period $T = 0.05$ With. Write it down $x(n)$.

2. On entrance discrete integrator, described equation (8.3), filed impact $x(n) = \delta(n)$. Find $y(n)$, if $y(0) = 0$ And $T = 0.5$ With.

3. On entrance discrete differentiator, described equal - niy (8.15), filed stepped impact $x(n) = 1(n)$. Find $y(n)$, If $T = 0.25$ With.

4. Draw a block diagram of a discrete integrator that implements integration using the rectangle (trapezoid) method .

5. Write it down difference equation discrete integrator, realizing integration By method rectangles, If on entrance integrator filed signal $x(t) = 2t$ And $T = 0.2$ With.

6. Write it down difference equation discrete integrator, implementing integration using the trapezoid method , if the input of the integral is

Torah the signal has been given $x(t) = 1(t)$ And $T = 0.5$ With .

Practical lesson No. 9

SMOOTHING OF DISCRETE SEQUENCES

9.1. Target work

At research real processes, How rule, instead of true-
 Noah physical quantities is registered random the value of $x(t)$,
 representing an additive mixture of the quantity f itself
 (t)

And interference $r(t)$, That There is $x(t) = f(t) + r(t)$. Interference $r(t)$
 Maybe generate-
 be located directly in the object being studied, get into it from the outside or
 be a random interference in the measurement and recording circuits.

The presence of interference in the sequence x
 (n) obtained
 V result discretization, makes it difficult receiving reliable info -

matsii about under study process. That's why sequence $x(n)$
 subjected to primary processing, purpose which is smoothing,
 That There is complete or partial elimination interference $r(n)$. Smoothing
 discrete sequence $x(n)$ is carried out using special algorithms.

The aim of the work is to study the algorithms for smoothing
 experimental data presented in the form of finite discrete sequences.

9.2. Main concepts And settlement formulas

The data smoothing algorithms studied in this paper are described by
 linear difference equations. Therefore, they can be considered How digital
 filters, which transform original By-
 sequence $x(n)$ V subsequence $y(n)$, being assessment
 useful component $f(n)$.

9.2.1. Smoothing sliding averaging

The essence of this smoothing method consists in the following
 average ordinate $x(n)$, $n = 0.1, \dots, N-1$, at intervals
 $[n-L/2, n+L/2]$, Where L - whole even number. Values smoothed
 sequences $y(n)$ are determined By formula

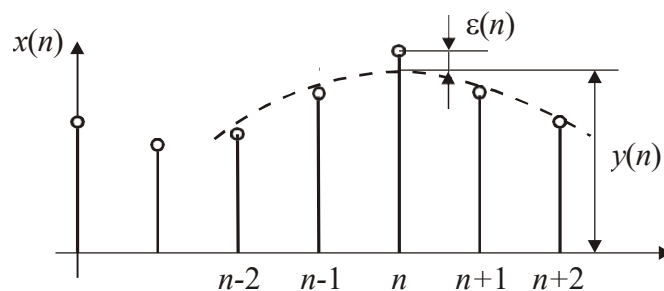
$$y(n) = \frac{1}{L+1} \sum_{\lambda=0}^L x(n+\lambda) \quad n = 0.1, \dots, N-L-1. \quad (9.1)$$

According to this formula value of $y(n+L/2)$ located How average arithmetic $L+1$ values $x(n), x(n+1), \dots, x(n+L)$. At this use are thinning out values, located left And on the right from the current issue discrete sequences. For example, at $L=4$ every element sequences $y(n)$ is calculated How average five values entrance- Noah sequences $x(n-2), x(n-1), x(n), x(n+1), x(n+2)$.

A filter that implements smoothing using the moving average method, significantly weakens harmonic components, frequencies which higher $\omega = 2\pi/(T \cdot L)$. Correct choice values L define- et quality departments high-frequency interference $r(n)$ from more low hour - totnaya component $f(n)$. Decrease L leads To insufficient alignment experimental data, A overstatement - To distortion significant features sequences $f(n)$. Because hour- tony spectra sequences $f(n)$ And $r(n)$ in advance unknown, size L usually are being picked up experimentally. Usually procedure anti-aliasing they are starting with values $L=2-4$ And increase in case necessity after analysis received results smoothing.

9.2.2. Smoothing fourth differences

Smoothing fourth differences is produced by way approx - simulations five neighboring values sequences $x(n)$ parabola With help method smallest squares. IN quality element evil eye - wife's sequences $y(n)$ accepted dot parabolas, the most the best way approximating values smoothed after - valency $x(n)$ V five points (rice. 9.1)



Rice. 9.1. Approximation sequences V five points

As shown using the least squares method, the value of this element is calculated by the formula

$$y(n) = x(n) - \varepsilon(n), \quad (9.2)$$

Where amendment $\varepsilon(n)$ proportional displaced reverse or direct differences fourth order:

$$\varepsilon(n) = \frac{1}{35} \Delta^4 x(n+2) - \frac{3}{35} \Delta^4 x(n-2). \quad (9.3)$$

Using the known relationships for calculating direct and inverse differences, it is easy to verify that

$$\varepsilon(n) = \frac{1}{35} [x(n-2) - 4 \cdot x(n-1) + 6 \cdot x(n) - 4 \cdot x(n+1) + x(n+2)]. \quad (9.4)$$

Substituting (9.4) into (9.2), we find a formula that allows us to calculate the ordinate of the output sequence:

$$y(n) = \frac{1}{35} [-3 \cdot x(n-2) + 12 \cdot x(n-1) + 17 \cdot x(n) + 12 \cdot x(n+1) - 3 \cdot x(n+2)]. \quad (9.5)$$

9.2.3. Exponential smoothing

Exponential smoothing is one of the simplest and most common methods for aligning sequences. It is based on the calculation of exponential means. Exponential smoothing of a sequence is carried out using a difference equation

$$y(n) = (1 - \alpha) \cdot y(n-1) + \alpha \cdot x(n), \quad (9.6)$$

Where α - constant coefficient ($0 < \alpha < 1$), called *constant smoothing*.

From expressions (9.6) follows, What current meaning smoothed By-sequences $y(n)$ equals previous her meaning plus some - paradise's share (α) the difference between the current value of the input sequence and the previous value of the smoothed output sequence.

If sequentially use ratio (9.6), That Exponential average $y(n)$ Can express through values input sequences $x(v)$, $v = 0, \dots, n$:

$$\begin{aligned} y(n) &= \alpha \cdot x(n) + (1 - \alpha) \cdot y(n-1) = \\ &= \alpha \cdot x(n) + \alpha \cdot (1 - \alpha) \cdot x(n-1) + (1 - \alpha)^2 \cdot y(n-2) = \dots \\ &= \alpha \sum_{v=0}^{n-1} (1 - \alpha)^v \cdot x(n-v) + (1 - \alpha)^n \cdot x(0). \end{aligned} \quad (9.7)$$

Like this in this way, the value of $y(n)$ turns out weighted sum everyone members sequences $x(n)$, and weights are falling exponentially V dependencies from deletions element input sequences.

If, For example, $\alpha = 0.3$, That current element sequences will have weight 0.3, A weights previous elements will make up respectively 0.21; 0.147; 0.1029 And etc. Constant anti-aliasing α accepts values from 0 to 1. Limit meaning $\alpha = 0$ corresponds occasion $L = \infty$ when smoothing by moving average. In this case, $y(n) = y(n-1)$. Limit meaning $\alpha = 1$ means, What previous values are not taken into account at all.

How shows practice, meaning constant anti-aliasing α follows accept V within from 0.01 to 0.3.

9.3. Methodological instructions

For research The smoothing algorithms described above generate a test discrete sequence

$$x(n) = f(n) + r(n), \quad n = 0, 1, \dots, N-1, \quad (9.8)$$

V which useful component $f(n)$ consists of from two harmonic sequences With various frequencies:

$$f(n) = \sin\left(\frac{2\pi}{M_1}n\right) + \cos\left(\frac{2\pi}{M_2}n\right), \quad (9.9)$$

and interference $r(n)$ represents by yourself centered random sequence - valence generated using standard functions MathCAD.

Centered random subsequence $r(n)$ is being formed as a difference

$$r(n) = r_1(n) - \text{mean}(r_1). \quad (9.10)$$

Here non-centered random subsequence is formed at help standard functions $\text{rnd}(x)$, That There is

$$r_1(n) = \text{rnd}(b), \quad (9.11)$$

where b is the upper limit of the scatter interval of random numbers,

A mean (r_1) – the average value determined by the system's means

MathCAD or directly By formula

$$\text{mean}(r_1) = \frac{1}{N} \sum_{n=0}^{N-1} r_1(n). \quad (9.12)$$

Thus, the test sequence finally takes the form

To evaluate the smoothing quality of various algorithms, the sum is used in the work

$$J = \sum_{n=-10}^N [y(n) - f(n)]^2 \quad (9.14)$$

The limits of variation of n in (9.14) are adopted such that when comparing smoothing algorithms with some reserve, the influence of the initial and final sections on which the smoothing algorithms do not work is excluded.

9.4. Work program

9.4.1. The main thing exercise

1. Form useful And random components smoothing - my sequences $x(n)$, $n = 0, 1, \dots, N-1$ ($N \approx 200$), having accepted their

parameters from Table 9.1 according to the given option. Observe the useful and random components, as well as the smoothed sequence as a whole.

Table 9.1

Parameters	Option numbers							
	1	2	3	4	5	6	7	8
b	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3
M_1	37	41	43	47	49	37	41	47
M_2	19	23	29	19	23	29	19	23

2. Compose program anti-aliasing sequences $x(n)$ By method sliding averaging And calculations values criterion (9.14), characterizing quality smoothing. Observe smoothed sequences And calculate meaning criteria at $L = 2, 4, 6, 8$. Values of the criterion for different L enter into the table and evaluate the influence of the parameter L on the quality of anti-aliasing.

3. Compose program anti-aliasing sequences $x(n)$ By method fourth differences. Observe smoothed sequence and calculate the value of the quality criterion.

4. Compose program exponential anti-aliasing after - valency $x(n)$. Observe smoothed out subsequence And

calculate values criteria quality at $\alpha = 0.1, 0.2, \dots, 0.9$. Data

bring in V table And build schedule dependencies criteria anti-aliasing quality from coefficient α Define optimal meaning ko-coefficient α .

5. Compare values criteria quality smoothing, obtained for various smoothing algorithms, and draw conclusions.

6. Write down transmission functions $H(z)$ And frequency transmission - functions $H(e^{j\omega T})$ for all smoothing algorithms studied in the work.

7. Compose programs calculation frequency response And phase response everyone studied filters anti-aliasing V scale relative frequencies $r = \omega T / 2\pi$ at $0 \leq r \leq 0.5$. Build respectively frequency response And phase response everyone filters in one figure. Compare the studied anti-aliasing filters by frequency properties .

9.5. Control questions And tasks

7. IN what kind cases recommended use procedure smoothing of experimental data?

8. Why at smoothing sliding averaging increase L leads To distortion useful component?

9. How will change frequency response algorithm sliding averaging at increase L ?

10. Please explain geometrically idea anti-aliasing fourth differences .

11. is least squares approximation ?

12. Show that the filter equations obtained through the fourth-order backward and forward differences are equivalent.

13. Find static coefficient transmissions filter, implementing smoothing by fourth differences.

14. Why must the condition in equation (9.6) be satisfied?
 $0 < \alpha < 1$?

15. Find static coefficient transmissions exponential filter.

16. How are changing frequency response And phase response exponential filter when changing α ?

17. Which algorithms smoothing, except considered, To you are known ?
f T —

10.1. Methodological instructions

In this paper, second-order digital filters are investigated. The transfer functions of the filters under study are determined using the method described above. All studies are conducted using the obtained transfer functions.

) of digital filters are calculated using the transfer function and are constructed taking into account their periodicity in the frequency range $[0, \pi / T]$.

To test the filtering properties of a digital filter, an input discrete sequence is generated

$$x(n) = \cos(\omega_1 T \cdot n) + \cos(\omega_2 T \cdot n), \quad n = 0, 1, \dots, N-1 \quad (N \approx 500), \quad (10.11)$$

Where meaning ω_1 is selected from stripes transmission filter, A meaning ω_2 - from stripes delays filter.

Day off signal $y(n)$ digital filter is calculated By once-nostny equation, received from transmission functions.

10.2. Work program

10.2.1. The main thing exercise

1. Having used V quality prototype standardized transmission function second order (cm. application P.2) And given V Table 10.2 type filter And meaning frequencies cut ω_s , find transfer function of the digital low - pass filter.

Table 10.1

Type And parameters		Numbers options							
		1	2	3	4	5	6	7	8
T	s	0.008	0.01	0.012	0.02	0.025	0.04	0.05	0.08
ω_s	refuse/s	150	120	100	80	40	50	30	25
FNC	—	Bat	Cheb2	Bat	Cheb1	Bat	Cheb3	Bat	Cheb2
HPF	-	Cheb1	Bat	Cheb3	Bat	Cheb2	Bat	Cheb1	Bat

2. Create a program to calculate the transfer function coefficients digital Low-pass filter. Calculate coefficients transmission functions filter For given V Table 10.2 values ω_s And T .

3. Compose program calculation frequency response And phase response digital

low-pass filter at
 $\omega \in [0, \pi / T]$. Build frequency response And phase response digital filter.

4. Form a discrete sequence according to (10.11) $x(n)$, $n = 0, 1, \dots, N-1$, ($N \approx 500$).

5. Using the previously found transfer function, write the difference equation of the digital filter and make up program for calculating discrete sequences $y(n)$ digital filter at input $x(n)$. Observe on screen discrete sequentially $x(n)$, $y(n)$ And do conclusion of work filter.

6. Repeat points 1–5 For given digital HPF.

7. Reduce the value of the sampling period T by 20%. Using the compiled program, calculate the transfer coefficients functions, frequency response And phase response digital low-pass filter And HPF. Compare frequency response and phase response for two values of the sampling period and draw a conclusion.

10.3. Control questions And tasks

1. Give it to me concept digital filter.
2. What is the difference between the division of digital filters according to the features “non-recursive and recursive” and “finite and infinite impulse responses”?
3. What methods are used to convert the transfer function of an analog prototype filter into the transfer function of a digital filter?
4. What conditions must the procedure for converting the transfer function of an analog prototype into the transfer function of a digital filter satisfy?
5. Please explain basic bilinear transformations.
6. Explain the meaning of the operation of denormalizing the transfer function of the low-pass filter.
7. Explain the meaning of the operation of transforming the transfer function of the low-pass filter.
8. How do the frequency responses of Butterworth and Chebyshev filters differ in the passband ?
9. How conditioned by restrictions on lower And upper limits of variation of the sampling period T of a digital filter?
10. Explain reason periodicity frequency digital filter characteristics.

Practical lesson No. 11

STUDY NON-RECURSIVE DIGITAL LOW-PASS FILTERS

11.1. Target work

One of the most important features of non-recursive digital filters is that, with the appropriate choice of parameters, they can have a strictly linear phase-frequency response (PFR). This property is, for example, decisive if it is necessary to ensure undistorted signal conversion. Non-recursive digital filters are also used in cases where the requirements cannot be met using Butterworth and Chebyshev filters, for example, to perform differentiation and integration of signals.

The aim of the work is to study the features of non-recursive digital low-pass filters with a linear phase response, master the method of calculating its coefficients using the weighting method and study the influence of the order N and the sampling period T on the frequency characteristics of the filter.

11.2. Main concepts And settlement formulas

11.2.1. Non-recursive digital low-pass filter With linear phase response

Non-recursive digital filters are described by the difference equation

$$y(n) = \sum_{v=0}^{N-1} b_v x(n-v). \quad (11.1)$$

Transfer function, received as a result applications direct z -transformations to equation (11.1) has the form

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{v=0}^{N-1} b_v z^{-v}. \quad (11.2)$$

How it is known, transmission function digital filter May be defined How direct z -transformation pulsed characteristics - sticks $h(n)$, That There is

$$H(z) = Z \left\{ h(n) \right\} = \sum_{v=0}^{N-1} h(n) z^{-v}. \quad (11.3)$$

It follows that the coefficients of the transfer function (difference equation) and the impulse response readings of non-recursive digital filters coincide.

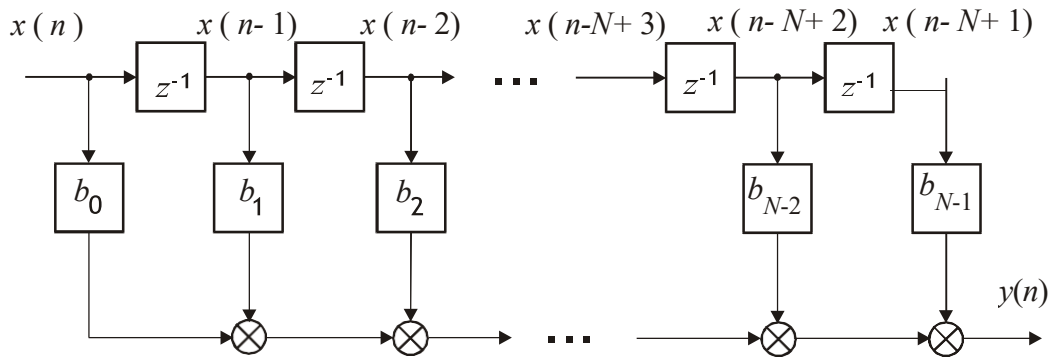
If V difference equation (11.1) produce replacement $b_v = h(v)$, $v = 0, 1, \dots, N-1$, That we will receive

$$y(n) = \sum_{v=0}^{N-1} h(v) x(n-v). \quad (11.4)$$

This There is Not What other, How convolution functions $x(n)$ And $h(n)$. Hence,

V case non-recursive digital filters difference equation and the convolution equation coincide.

Algorithm functioning non-recursive digital filters is presented in the form of a structural diagram shown in Fig. 11.1.



Rice. 11.1. Structural scheme non-recursive digital filter

The frequency transfer function of non-recursive digital filters is determined by the expression

$$H(e^{j\omega T}) = \sum_{v=0}^{N-1} h(v) e^{-j\omega T v}. \quad (11.5)$$

Non-recursive digital filters with linear phase response, depending on the number N and the type of symmetry of the impulse response readings are divided into four types. Two of them are suitable for implementing low-pass filters with a linear phase response. The frequency transfer functions of these filters have the following form:

1) For type 1 (N – odd, symmetrical coefficients $b_v = b_{N-v-1}$):

$$H_1(e^{j\omega T}) = e^{-jK} \sum_{m=0}^K c_m \cos(m\omega T) \quad (11.6)$$

$$K = (N-1)/2; \quad c_0 = b_K; \quad c_m = 2b_{K-m}, \quad m=1,2,\dots,K;$$
 where

2) for type 2 (N – even, symmetric coefficients $b_v = b_{N-v-1}$):

$$H_2(e^{j\omega T}) = e^{-j(K+0.5)\omega T} \sum_{m=0}^K c_m \cos((m+0.5)\omega T), \quad (11.7)$$

Where $K = (N-1)/2$; $c_m = 2b_{K-m}$, $m = 0, 1, \dots, K$.

These filters contain only zeros, so to obtain a frequency response with a steep cutoff, it is necessary to take a large N . Therefore, the implementation of filters requires a large number of delay elements. These filters are always stable.

11.2.2. Calculation coefficients non-recursive Low-pass filter by weighting method

for calculating the coefficients of non-recursive digital filters. One of the most commonly used is the weighting method. Here, the requirements for the frequency response of a digital low-pass filter are specified as a function.

$$H_d(e^{j\omega T}) = \begin{cases} e^{-j\omega T(N-1)/2} & \text{at } |\omega| \leq \omega_s, \\ 0 & \text{in others cases.} \end{cases} \quad (11.8)$$

According to (11.8), the desired frequency response and phase response of the filter are determined respectively by the formulas:

$$H_d(\omega) = \begin{cases} H_d(e^{j\omega T}) & \text{at } |\omega| \leq \omega_s, \\ 0 & \text{V others cases.} \end{cases} \quad (11.9)$$

$$\varphi(\omega) = \arg H_d(e^{j\omega T}) = -\omega(N-1)T/2. \quad (11.10)$$

Pulse characteristics $h_d(n)$ Can get by way calculated - leniya coefficients rows Fourier functions (11.8), continued periodically With period $\omega_p = 2\pi/T$:

$$h_d(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega T(n-(N-1)/2)} d\omega \quad (11.11)$$

After transformations (11.11) we will have

$$h_d(n) = \frac{\sin[\omega_c(n-(N-1)/2)T]}{\pi(n-(N-1)/2)}, \quad n = 0, 1, \dots \quad (11.12)$$

Obviously, $h_d(n)$ has endless length. One from possible

One way to obtain a non-recursive digital filter is to truncate infinite rows (11.12) by way discards counts, corresponding -

existing $n \geq N$, That There is V quality pulsed characteristics filter accepted function

$$h(n) = h_d(n), \quad 0 \leq n \leq N-1. \quad (11.13)$$

However, simple truncation of the function (11.12) does not give good results due to the Gibbs phenomenon. The amplitude frequency response of the filter when using simple truncation has high-level spikes and pulsations before and after the discontinuity point of the approximated frequency response. For example, when approximating the ideal characteristic of the low-pass filter (11.8), the maximum amplitude of the pulsations in the frequency response is about 9 %, and with an increase in the length of the pulse response it does not decrease.

Wide common way elimination harmful influences phenomena Gibbs consists of in weighing functions $h_d(n)$ at help - cabbage soup functions $w(n)$. Pulse characteristics non-recursive digital - ditch filter find V view

$$h(n) = h_d(n) w(n), \quad (11.14)$$

and discrete function $w(n)$, given on interval $[0, N-1]$,

they call window function (window). By the way, simple truncation functions $h_d(n)$ equivalent application rectangular window

$$w(n) = 1, \quad 0 \leq n \leq N-1. \quad (11.15)$$

The most frequently used window functions are Bartlett, Hann, Hamming, and Blackman (Appendix P.3). The use of window functions leads, firstly, to a reduction in the sidelobe level and, as a consequence, to less ripple in the frequency response in the passband and better suppression in the filter's stopband. Secondly, the window function increases the width of the main lobe of the frequency response.

11.3. Methodological instructions

The work program includes the calculation of the pulse and frequency characteristics of a non-recursive digital low-pass filter, and their comparison for different values of the filter order. N and the sampling period T , and also calculation of the filter response for a given input signal.

Impulse response of filter $h_d(n)$ is calculated by formula (11.12). At this necessary remember, What at $n = (N-1)/2$ Maybe to manifest itself as an uncertainty of type 0/0 To eliminate this uncertainty, the if conditional statement should be used.

The frequency characteristics of a digital filter are calculated using the formula (11.5) And With taking into account their periodicity V interval frequencies $[0, \pi / T]$.

When preparing a report, the characteristics corresponding to different values of N and T , it is recommended to build on one figure for ease of comparison.

Calculation day off sequences filter $y(n)$ at given entrance impact $x(n)$ is carried out according to formula (11.4).

11.4. Work program

11.4.1. The main thing exercise

1. Calculate the impulse response of a non-recursive digital low-pass filter with a linear phase response, using the parameters as initial data filter, given V table 11.1, And formula (11.12). Order filter $N = 21$.

Table 11.1

Parameters		Number options							
		1	2	3	4	5	6	7	8
T	c	0.008	0.01	0.012	0.02	0.025	0.04	0.05	0.08
ω_c	rad/s	120	120	100	80	50	40	30	25
Window function		Bartlett		Henna		Hamming		Blackman	

2. Compose calculation program pulsed characteristics $h(n)$ non-recursive digital filter With help rectangular windows (simple truncations $h_d(n)$) at $N = 21$.

3. Compose program calculation pulsed characteristics $h(n)$ non-recursive digital filter With help given V table 11.1 window functions $w(n)$ at $N = 21$.

Note. Analytical expressions window functions $w(n)$ given V application P.3.

4. Calculate and plot impulse characteristics on one figure non-recursive filters corresponding to a rectangular and specified window option.

5. Create a program for calculating the frequency characteristics of non - recursive digital filters, corresponding rectangular And for-

this window option. Construct and compare the frequency response and phase response of these filters .

6. Form discrete sequence $x(n)$,
consisting of two sinusoidal components

$$x(n) = \sin(\omega_1 T \cdot n) + \sin(\omega_2 T \cdot n), \quad n = 0, 1, \dots, 500 \quad (11.16)$$

having chosen values ω_1 And ω_2 V strip transmission filter. Observe this sequence.

7. Calculate the discrete sequence $y(n)$ at the output filter at entrance impact $x(n)$. Observe discrete By-sequences $x(n), y(n)$ And do conclusion about distortion You-running sequences By form.

8. Accept $N = 31$ And With help compiled programs calculate - become impulsive and frequency characteristics filter, received by using given window functions. Build their together With one-registered characteristics filter With $N = 21$ And compare.

9. Change meaning period discretization T on 20% V that And friend - I'm talking side at $N = 31$ And With help compiled programs calculate - Find the impulse and frequency responses of a filter obtained using a given window function. Plot them together with the same-named filter characteristics obtained above and compare them.

11.5. Control questions And tasks

1. Write down the conditions for undistorted reproduction of a signal by a linear stationary system.
2. How differ non-recursive filters With symmetrical and antisymmetric impulse responses?
3. Explain the differences between causal and non-causal digital filters .
4. Why non-recursive digital filters Always stable?
5. expression (11.5) as the initial one .
6. How looks like amplitude-phase frequency characteristic of non-recursive digital filter?
7. List famous To you methods calculation non-recursive digital filters.
8. Please explain characteristic peculiarities window functions.

