

MINISTRY OF SCIENCE AND HIGHER EDUCATION OF THE RUSSIAN FEDERATION
FEDERAL STATE AUTONOMOUS EDUCATIONAL INSTITUTION OF HIGHER EDUCATION
"NORTH CAUCASUS FEDERAL UNIVERSITY"

SUGGESTIONS FOR TEACHING
for the performance of practical work
in the discipline "**Higher Mathematics/Advanced Mathematics**"
for students

Areas of training	11.03.02 Infocommunication Technologies and Communication Systems
Profile	Intelligent Data Processing in Infocommunication Systems and Networks

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PRACTICAL LESSON PLANS

1 semester

SECTION I. INTRODUCTION TO THE ANALYSIS

Lesson No 1. Sets. Method of mathematical induction. Bernoulli's inequality

The purpose of the lesson: to determine the content of the concept of "set", to learn how to perform various operations on sets, to work out the classification of numerical sets. To get acquainted with the method of mathematical induction used in proving mathematical statements, to use Bernoulli's inequality in proving inequalities.

Issues for discussion:

1. Definition of a set, operations on sets, their representation using Euler-Venn diagrams.
2. Numerical sets.
3. The method of mathematical induction, Bernoulli's inequality.

Tasks solved in the classroom

Example 1. Solve inequality $|x+3|+4x > |2x-8|$.

Since the meaning of the expression of the module under the sign changes its sign during the transition from positive to negative values, we will find these critical points: $x+3=0$, $2x-8=0$, therefore, $x_1=-3$, $x_2=4$. Let us divide the numerical line with these points into intervals $(-\infty-3]$, $(-3,4)$, $[4,+\infty)$ and at each interval reveal the moduli in the initial inequality by definition:

$$\begin{aligned}
 1). & \begin{cases} x \in (-\infty-3], \\ -x-3+4x > -2x+8 \end{cases} \begin{cases} x \in (-\infty-3], \\ 5x > 11 \end{cases} \begin{cases} x \in (-\infty-3], \\ x > 2,2 \end{cases} x \in \emptyset; \\
 2). & \begin{cases} x \in (-3,4], \\ x+3+4x > -2x+8 \end{cases} \begin{cases} x \in (-3,4], \\ 7x > 5 \end{cases} \begin{cases} x \in (-3,4], \\ x > \frac{5}{7} \end{cases} x \in \left(\frac{5}{7}, 4\right]; \\
 3). & \begin{cases} x \in (4,+\infty], \\ x+3+4x > 2x-8 \end{cases} \begin{cases} x \in (4,+\infty], \\ 3x > -11 \end{cases} \begin{cases} x \in (4,+\infty], \\ x > -\frac{11}{3} \end{cases} x \in (4,+\infty];
 \end{aligned}$$

The solution to this inequality is the values $x \in \left(\frac{5}{7}, +\infty\right]$ of .

Tasks for independent work

1. Using Euler-Venn diagrams, show the validity of the following statements:
 - a). $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$;
 - b). $(A \cup C) \setminus B = (A \setminus B) \cup (C \setminus B)$;
 - c). $(A \setminus B) \cup C = (A \setminus C) \setminus B$;
 - d). $(A \setminus B) \setminus C = (A \setminus C) \setminus B$
2. Specify by enumerating all the elements of the set defined using the following characteristic properties:
 - a). $A = \{x \in N | x \leq 5\}$;
 - b). $B = \{x \in N | x < 0\}$;

- c). $C = \{x \in \mathbb{Z} \mid |x| \leq 2\}$;
- d). $D = \{x \in \mathbb{Z} \mid (x-3)(x^2-1) = 0; \quad x \geq 0\}$
3. Define sets $A \cup B$; $A \cap B$; $A \setminus B$; $B \setminus A$ if
- a). $A = \{x \in \mathbb{Z} \mid |x| \leq 2\}$, $B = \{x \in \mathbb{Z} \mid x^2 - 5x < 0\}$;
- b). $A = \{x \in \mathbb{Z} \mid |x-2| < 1\}$, $B = \{x \in \mathbb{Z} \mid (-1; 2]\}$;
- c). $A = \{x \in \mathbb{Z} \mid |x-1| < 2\}$, $B = \{x \in \mathbb{Z} \mid |x-2| < 3\}$;
- d). $A = \{x \in \mathbb{R} \mid |x-1| < 2\}$, $B = \{x \in \mathbb{R} \mid |x-1| + |x-2| < 3\}$;
- e). $A = \{x \in \mathbb{R} \mid x^2 + x - 20 < 0\}$, $B = \{x \in \mathbb{R} \mid x^2 - 7x + 12 < 0\}$
4. Solve the equations
- a). $2|x+3| - 4|x+5| + |x-2| = 0$;
- b). $|x+2| - 3|x-1| + 5|x-4| = 0$;
- c). $|x+5| + 4|x+3| - |x-1| = 0$;
- d). $|x-1| + 5|x+5| - 3|x-2| = 4$

Tasks solved in the classroom

Example 2. To prove that the identity is true n for any natural:

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

- 1). Let's check the equality at $n=1$: $1^2 = \frac{n(n+1)(2n+1)}{6}$; $1 = \frac{6}{6} = 1$ - true.
- 2). Suppose that equality is true at $n=k$, that is $1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$, true.
- 3). Let us prove the truth of the equation and at $n=k+1$, that is, we shall prove

$$1^2 + 2^2 + \dots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Let's transform the left side of this equation by replacing the first k terms with the right part of the equation from point 2):

$$\begin{aligned} \frac{k(k+1)(2k+1)}{6} + (k+1)^2 &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{(k+1)(k(2k+1) + 6(k+1))}{6} = \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}. \end{aligned}$$

Thus, we have proved the equality of point 3), using the equality of point 2), according to the MMI, this statement is true $\forall n \in \mathbb{N}$.

Example 3. To prove that $2^n > 2n+1$ with any natural $n \geq 3$

- 1). With $n=3$ inequality, it is true: $2^3 > 2 \cdot 3 + 1$.
- 2). Suppose that $2^k > 2k+1$ ($k \geq 3$).
- 3). Let's prove that inequality is true $2^{k+1} > 2(k+1)+1$.

In fact, $2^{k+1} = 2 \cdot 2^k > 2 \cdot (2k+1) = (2k+3) + (2k-1)$. But $2k-1 > 0$ with any natural k . It follows $2^{k+1} > 2k+3$ that inequality $2^n > 2n+1$ is valid for all natural $n \geq 3$

Example 4. Prove that $(3^{2n+1} + 40n - 67) : 64$ for any natural n

- 1). If $n=1$, then $3^3 + 40 - 67 = 0$, and $0:64$.
- 2). Suppose that at $n = k$ $(3^{2k+1} + 40k - 67):64$.
- 3). Let us prove the truth of the statement with $n = k + 1$, that is, that $(3^{2k+3} + 40(k+1) - 67):64$

$$\begin{aligned} \text{In fact, } 3^{2k+3} + 40(k+1) - 67 &= 9 \cdot 3^{2k+1} + 40k + 40 - 67 = \\ &= 9 \cdot (3^{2k+1} + 40k - 67) - 8 \cdot (40k - 67) + 40 = 9 \cdot (3^{2k+1} + 40k - 67) - 320k + 576 = \\ &= 9 \cdot (3^{2k+1} + 40k - 67) + 64 \cdot (9 - 5k), \end{aligned}$$

But the first term is divisible by 64 on the assumption, and the second term is divisible by 64, since one of the factors is 64. Therefore the sum will be divided by 64. Thus we have proved that $(3^{2n+1} + 40n - 67):64$ for any natural number n .

Tasks for independent work

Using the method of mathematical induction, prove equations

$$\text{a). } 1 \cdot 2^2 + 2 \cdot 3^2 + \dots + (n-1) \cdot n^2 = \frac{n(n^2 - 1)(3n + 2)}{12};$$

$$\text{b). } 1^2 + 3^2 + \dots + (2n-1)^2 = \frac{n(4n^2 - 1)}{3};$$

$$\text{c). } 1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2} \right)^2, n \in N;$$

$$\text{d). } 1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{n-1} \cdot n^2 = (-1)^{n-1} \cdot \frac{n(n+1)}{2}$$

6. Using the method of mathematical induction, prove the following inequalities

$$\text{a). } 2^n > n^3, \forall n \geq 10;$$

$$\text{b). } 2^{n-1} > n(n+1) \quad \forall n \geq 7;$$

$$\text{c). } 4^n \geq 3^n + n^2 \quad \forall n \in N;$$

$$\text{d). } 3^n > 5n + 1 \quad \forall n \geq 3$$

7. Prove that for any natural n $9^{n+1} - 8n - 9$ is a multiple of 16.

8. Prove that for any natural n $5 \cdot 2^{3n-2} + 3^{3n-1}$ is a multiple of 19.

9. Prove that for any natural n $n(2n^2 - 3n + 1)$ is a multiple of 6.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 2. Function of a real variable

The purpose of the lesson: to determine the content of the concept of "real function of a real argument", the main properties of functions, their simplest classification.

Issues for discussion:

1. The simplest classification of functions of a real variable is: monotonic, even and odd, periodic, bounded and unbounded functions.
2. Ways to define functions.
3. Arithmetic operations on functions.
4. Basic elementary functions, their properties, graphs.

Tasks solved in the classroom

Example 5. Find the scope of the function definition $y = \sqrt{x^2 - 7x + 12} + \frac{2x - 3}{x - 5}$.

The domain of this function consists of those values x at which both terms take real values. Therefore,

$$\begin{cases} x^2 - 7x + 12 \geq 0 \\ x - 5 \neq 0. \end{cases}$$

Solving the quadratic inequality $x^2 - 7x + 12 \geq 0$, we obtain $x \leq 3$ and $x \geq 4$. Thus, the domain of definition of this function consists of a set of three intervals: $x \leq 3$; $x \geq 4$; $x \neq 5$ (Fig. 6).

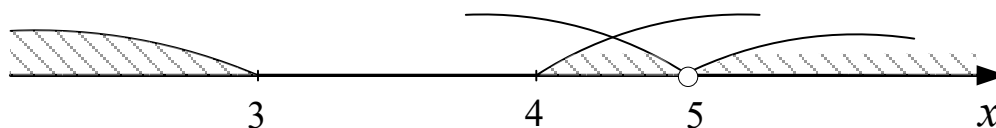


Fig. 6

Therefore, $D(y) = (-\infty; 3] \cup [4; 5) \cup (5; +\infty)$.

Example 6. Set the parity or oddness of the function $y = x^3 \cos 2x$.

$D(y) = (-\infty; +\infty)$, that is, the domain of the definition of the function is a symmetric set,

$f(x) = x^3 \cos 2x$, $f(-x) = (-x)^3 \cos 2(-x) = -x^3 \cos 2x$. Therefore,
 $f(-x) = -f(x)$ that is, this function is odd.

Tasks for independent work

Find the scope of a function definition

a). $y = 3x^3 + 5x^2 - 7x + 2$;

b). $y = \frac{1}{7x - 2}$;

c). $y = \frac{x - 1}{x^2 - 7x + 12}$;

d). $y = \frac{x^2 - 5x + 4}{x^2 + x + 1}$;

e). $y = \frac{5}{\sqrt{x - 4}}$;

f). $y = \frac{1}{\sqrt[3]{8 - x}}$;

g). $y = \sqrt{4 - x^2}$;

h). $y = \sqrt{\frac{x+1}{x-1}}$;

i). $y = \lg(x^3 - 3)$;

j). $y = \sin(2x + 3)$;

k). $y = \operatorname{tg} 2x$;

m). $y = \arcsin(5x - 8)$

n). $y = \frac{1}{\sqrt{-x^2 + x + 2}} + \lg(x - 1)$;

o). $y = \log_2(x - 1) + x^2$;

p). $y = \sqrt{5 - x} + \sqrt{x + 3}$;

p). $y = 2x^2 + \lg(x - 1) + \frac{1}{x - 3}$;

c). $y = \frac{\sin x}{\lg(x^2 - 4)}$;

t). $y = \lg \frac{x^2 - 5x + 6}{x^2 + x + 1}$;

s). $y = \frac{1}{\lg(1 - x)} + \sqrt{x + 2}$;

f). $y = \sqrt{3 - x} + \arcsin \frac{3 - 2x}{5}$;

x). $y = \arcsin \frac{x - 3}{2} - \lg \sqrt{4 - x}$;

h). $y = \arccos \frac{x - 4}{x}$

11. Find the Function Value Area

a). $y = \frac{1}{2 - \cos 3x}$; b). $y = \frac{x^2 + 8}{x}$; ;

c). $y = \frac{x + 2}{x - 3}$; d). $y = \frac{x^2 - 4}{x - 1}$. .

12. Which of the following functions are even, which are odd, which are neither even nor odd

a) $y = x^2 - 2x^2$;

b) $y = x - x^2$;

c) $y = \cos x$;

d) $y = x - \frac{x^2}{6} + \frac{x^5}{120}$;

e) $y = \sin x - \cos x$;

f) $f(x) = x + \sin x$;

g) $y = \frac{x^2}{\sqrt{3x^2 - 2}}$;

h) $q(x) = \sqrt{1 - x^2}$;

i) $s(t) = \lg \frac{1 - t}{1 + t}$;

j) $f(x) = x^3, x \in [-4; 7]$.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 3. Calculating the Sequence Limit

The purpose of the lesson: to consider the basic concepts of the theory of numerical sequences and their limits, to learn how to calculate the limits of numerical sequences.

Issues for discussion:

1. The limit of the numerical sequence and its geometric meaning.
2. Properties of converging sequences.
3. Infinitely small and infinitely large sequences, the connection between them.
4. Calculating the limits of a numerical sequence: uncertainty of the form $\left(\frac{\infty}{\infty}\right), (\infty - \infty)$.
5. Number e and natural logarithms.

Tasks solved in the classroom

Example 7. The sequence $\{(-1)^{n+1}\} = \{1, -1, 1, -1, \dots\}$ does not tend to any limit.

In fact, let us assume that this sequence has a limit equal to the number a . Let $\varepsilon = \frac{1}{3}$

Consider the vicinity of this point $\left(a - \frac{1}{3}, a + \frac{1}{3}\right)$.

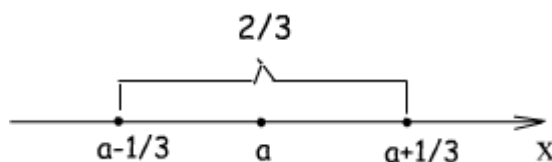


Fig. 21

Its length is $2/3$, i.e. $\rho\left(-\frac{1}{3}; \frac{1}{3}\right) = \left|-\frac{1}{3}, -\frac{1}{3}\right| = \frac{2}{3}$. Obviously, this neighborhood cannot contain both point 1 and point (-1) at the same time, because $\rho(-1; 1) = 2$, a $2 > 2/3$. For the sake of certainty, let us assume that point 1 does not belong to our neighborhood. But $x_n = 1$ with, $n = 1, 3, 5, \dots$ that is, outside our neighborhood there are an infinite number of elements of the sequence.

Thus, a point a cannot be the limit of our sequence, and since this point is arbitrary, this sequence has no limit, that is, $\lim_{n \rightarrow \infty} (-1)^{n-1} = \infty$.

Example 8. Prove that the sequence $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \dots, \frac{n}{2n+1}, \dots$ at $n \rightarrow \infty$ has the limit of the number $\frac{1}{2}$.

Let ε be an arbitrary positive number. It is necessary to prove that there exists such a number $N = N(\varepsilon)$ that when the $n > N$ inequality holds $\left|u_n - \frac{1}{2}\right| < \varepsilon$.

Let us make the difference modulo $\left|\frac{n}{2n+1} - \frac{1}{2}\right| = \left|-\frac{1}{2(2n+1)}\right| = \frac{1}{2(2n+1)}$. Thus, the inequality $\left|u_n - \frac{1}{2}\right| < \varepsilon$ holds if $\frac{1}{2(2n+1)} < \varepsilon$, from where $2(2n+1)\varepsilon > 1$ or $n > \frac{1}{4\varepsilon} - \frac{1}{2}$. It

is clear that an *integer part of the number*, i.e., which $\frac{1}{4\varepsilon} - \frac{1}{2}$ was to be proved, $N = \left[\frac{1}{4\varepsilon} - \frac{1}{2} \right]$ can be taken as the number N .

Let us give $\varepsilon = \frac{1}{80}$, then $n > \frac{1}{4 \cdot \frac{1}{80}} - \frac{1}{2} = 19\frac{1}{2}$. Therefore, starting from the number $n = 20$

, the inequality, that is $\left| u_n - \frac{1}{2} \right| < \varepsilon$, which means $\left| \frac{n}{2n+1} - \frac{1}{2} \right| < \frac{1}{80}$, will be fulfilled

$\lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2}$. From a geometric point of view, this means that $\left(\frac{1}{2} - \frac{1}{80}, \frac{1}{2} + \frac{1}{80} \right)$ the first 19 terms of the sequence are outside $u_1, u_2, \dots, u_{19}$ the interval, and starting from the twentieth, all the terms of the sequence will fall within the specified interval.

Example 9. The sequence $x_n = (-1)^n$ is constrained because $|x_n| = 1$, but is divergent because it has no limit.

Tasks for independent work

1. Using the definition of the limit of the numerical sequence, to prove that

$$\begin{array}{ll} \text{a). } \lim_{n \rightarrow \infty} \frac{3n-2}{3n-1} = 1; & \text{b). } \lim_{n \rightarrow \infty} \frac{2n-5}{3n+1} = \frac{2}{3}; \\ \text{c). } \lim_{n \rightarrow \infty} \frac{2-3n}{n+1} = -3; & \text{d). } \lim_{n \rightarrow \infty} \frac{1-3n^2}{n^2+2} = -3. \end{array}$$

2. Using the definition of the limit of the sequence, prove that $\lim_{n \rightarrow \infty} u_n = a$. Give the number from which all the terms of the sequence $\{u_n\}$ are in ε the vicinity of point a , if

$$\begin{array}{ll} \text{a). } u_n = \frac{6n+7}{11-2n}, & a = -3, \quad \varepsilon = 0,001; \\ \text{b). } u_n = \frac{3n^2+2}{5n^2+4}, & a = \frac{3}{5}, \quad \varepsilon = 10^{-4} \end{array}$$

Tasks solved in the classroom

Example 10. Calculate the limit of each numerical sequence

$$\begin{array}{l} 1. \lim_{n \rightarrow \infty} \frac{35n^4 + 7n^3 - 4n + 2}{5n^4 + 3n^2 - 150n - 4} = \left(\frac{\infty}{\infty} \right) = \lim_{n \rightarrow \infty} \frac{35 + \frac{7}{n} - \frac{4}{n^3} + \frac{2}{n^4}}{5 + \frac{3}{n^2} - \frac{150}{n^3} - \frac{4}{n^4}} = \frac{35}{5} = 7; \\ 2. \lim_{n \rightarrow \infty} \frac{25n^4 + 5n^3 - 7n + 4}{n^5 + n - 9} = \left(\frac{\infty}{\infty} \right) = \lim_{n \rightarrow \infty} \frac{\frac{25}{n} + \frac{5}{n^2} - \frac{7}{n^4} + \frac{4}{n^5}}{1 + \frac{1}{n^4} - \frac{9}{n^5}} = \frac{0}{1} = 0; \\ 3. \lim_{n \rightarrow \infty} \frac{30 - n + 8n^2 + 5n^5}{15n + 4n^2 + n^3} = \left(\frac{\infty}{\infty} \right) = \lim_{n \rightarrow \infty} \frac{\frac{30}{n^5} - \frac{1}{n^4} + \frac{8}{n^3} + 5}{\frac{15}{n^4} + \frac{4}{n^3} + \frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{5}{\downarrow 0} = \infty. \end{array}$$

From the above solution, it should be concluded:

1. If the major powers in the numerator and denominator are equal, then the limit is equal to the ratio of the coefficients for the higher powers;
2. If the power of the numerator is less than the power of the denominator, then the limit is 0;
3. If the power of the numerator is greater than the power of the denominator, then the limit is ∞ .

Example 11. Calculate $\lim_{n \rightarrow \infty} \frac{(n-1)! + (n+1)!}{(n+2)!}$.

Let's use the property $n! = 1 \cdot 2 \cdot \dots \cdot (n-1)n$, then the numerator and denominator can be reduced to the form:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{(n-1)! + (n+1)!}{(n+2)!} &= \lim_{n \rightarrow \infty} \frac{(n-1)!(1 + n(n+1))}{(n-1)!n(n+1)(n+2)} = \lim_{n \rightarrow \infty} \frac{1 + n(n+1)}{n(n+1)(n+2)} = \\ &= \lim_{n \rightarrow \infty} \frac{1}{n(n+1)(n+2)} + \lim_{n \rightarrow \infty} \frac{n(n+1)}{n(n+1)(n+2)} = 0 + \lim_{n \rightarrow \infty} \frac{1}{n+2} = 0 \end{aligned}$$

Example 12. Calculate $\lim_{n \rightarrow \infty} (\sqrt{n(n-3)} - (n+2))$.

Let's get rid of the difference of infinitely large quantities, for this we multiply and divide by the expression conjugate to the given, thus obtaining the difference of squares

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{(\sqrt{n(n-3)} - (n+2))(\sqrt{n(n-3)} + (n+2))}{\sqrt{n(n-3)} + n + 2} &= \lim_{n \rightarrow \infty} \frac{(n(n-3) - (n+2)^2)}{\sqrt{n(n-3)} + 2 + n} = \lim_{n \rightarrow \infty} \frac{n^2 - 3n - n^2 - 4n - 4}{\sqrt{n(n-3)} + 2 + n} = \\ &= \lim_{n \rightarrow \infty} \frac{-7n - 4}{\sqrt{n(n-3)} + 2 + n} = \lim_{n \rightarrow \infty} \frac{-7 - \frac{4}{n}}{\sqrt{1 - \frac{3}{n} + \frac{2}{n}} + 1} = -\frac{7}{2} \end{aligned}$$

Example 13. Calculate the limit $\lim_{n \rightarrow \infty} \left(\frac{2n+3}{2n-1} \right)^{3n}$.

Let us use the limit $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e$, since, substituting the limit value of n , we will have an uncertainty of the form (1^∞) , we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{2n+3}{2n-1} \right)^{3n} &= (1^\infty) = \lim_{n \rightarrow \infty} \left(\frac{2n-1+4}{2n-1} \right)^{3n} = \lim_{n \rightarrow \infty} \left(1 + \frac{4}{2n-1} \right)^{3n} = \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{4}{2n-1} \right)^{\frac{2n-1}{4} \cdot \frac{4}{2n-1} \cdot 3n} = e^{\lim_{n \rightarrow \infty} \frac{4}{2n-1} \cdot 3n} = e^6. \end{aligned}$$

Tasks for independent work

1. Find $\lim_{n \rightarrow \infty} u_n$ if

a). $u_n = \frac{n}{n^2 + 1}$;

b). $u_n = \frac{2 + 4 + 6 + \dots + 2n}{n + 2}$;

c). $u_n = \frac{(n+3)!}{2(n+1)! - (n+2)!}$;

d). $u_n = \left(n - \sqrt[3]{n^3 - n^2} \right)$.

2. Calculate the limit of the numerical sequence:

a). $\lim_{n \rightarrow \infty} \frac{4 + n!}{(n+2)!}$;

e). $\lim_{n \rightarrow \infty} \frac{n^2 - 5}{7 + n\sqrt{n}}$;

$$\text{b). } \lim_{n \rightarrow \infty} \frac{n(n-2)(n-1)}{n^4 + 5n + 3};$$

$$\text{c). } \lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - \sqrt{n^2 - n});$$

$$\text{d). } \lim_{n \rightarrow \infty} \left[\frac{1 + 2 + \dots + n}{n + 1} - \frac{n}{6} \right];$$

$$\text{f). } \lim_{n \rightarrow \infty} \left(n + \sqrt[3]{1 - n^3} \right);$$

$$\text{g). } \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{4} + \frac{1}{16} + \dots + \frac{1}{4^n}}{1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n}}$$

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 4. Calculating the Limit of a Function

The purpose of the lesson: to get acquainted with the basic concepts of the theory of function limits, to learn how to calculate the limits of functions, to use the rule of uncertainty disclosure.

Issues for discussion:

1. The limit of a function at a point and its properties.
2. A rule for disclosing uncertainty when calculating the limit of a function.

Tasks solved in the classroom

Example 14. Calculate the limit of the function $\lim_{x \rightarrow 2} \frac{x^2 + x - 2}{x^2 + 1}$.

Since $\lim_{x \rightarrow 2} (x^2 + 1) = 5 \neq 0$, we apply the limit theorem of the quotient

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 2}{x^2 + 1} = \frac{4 + 2 - 2}{4 + 1} = \frac{4}{5}.$$

Example 15. Calculate the limit of the function $\lim_{x \rightarrow 1} \frac{x^2 - 2x + 1}{x^2 - x}$.

Let's use the rule, decomposing the numerator and denominator of a fraction into factors, we

$$\text{get } \lim_{x \rightarrow 1} \frac{x^2 - 2x + 1}{x^2 - x} = \left(\frac{0}{0} \right) = \lim_{x \rightarrow 1} \frac{(x-1)^2}{x(x-1)} = \lim_{x \rightarrow 1} \frac{x-1}{x} = \frac{0}{1} = 0.$$

Example 16. Calculate the limit of the function $\lim_{x \rightarrow 2} \frac{x^2 - 6x + 8}{x^2 - 8x + 12}$.

$$\lim_{x \rightarrow 2} \frac{x^2 - 6x + 8}{x^2 - 8x + 12} = \left(\frac{0}{0} \right) = \lim_{x \rightarrow 2} \frac{(x-2)(x-4)}{(x-2)(x-6)} = \lim_{x \rightarrow 2} \frac{x-4}{x-6} = \frac{2}{4} = \frac{1}{2}.$$

Example 17. Calculate the limit of the function $\lim_{x \rightarrow 1} \frac{x^3 - 6x^2 + 11x - 6}{x^2 - 3x + 2}$.

$$\lim_{x \rightarrow 1} \frac{x^3 - 6x^2 + 11x - 6}{x^2 - 3x + 2} = \left(\frac{0}{0} \right) = \lim_{x \rightarrow 1} \frac{(x-1)(x-2)(x-3)}{(x-1)(x-2)} = -2.$$

Example 18. Calculate the limit of the function $\lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - \sqrt{1-x+x^2}}{x^2 - x}$.

Let us multiply the numerator and denominator of the fraction by the expression conjugate the

$$\begin{aligned} \text{numerator } \lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - \sqrt{1-x+x^2}}{x^2 - x} &= \left(\frac{0}{0} \right) = \lim_{x \rightarrow 0} \frac{1+x+x^2 - 1+x-x^2}{x(x-1)(\sqrt{1+x+x^2} + \sqrt{1-x+x^2})} = \\ &= \lim_{x \rightarrow 0} \frac{2x}{x(x-1)(\sqrt{1+x+x^2} + \sqrt{1-x+x^2})} = \frac{2}{-1 \cdot (1+1)} = -1. \end{aligned}$$

Example 19. Calculate the limit of the function $\lim_{x \rightarrow 1} \frac{\sqrt{x+1} - \sqrt{2}}{\sqrt{x+3} - 2}$.

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\sqrt{x+1} - \sqrt{2}}{\sqrt{x+3} - 2} &= \lim_{x \rightarrow 1} \frac{(\sqrt{x+1} - \sqrt{2})(\sqrt{x+1} + \sqrt{2})(\sqrt{x+3} + 2)}{(\sqrt{x+3} - 2)(\sqrt{x+3} + 2)(\sqrt{x+1} + \sqrt{2})} = \\ &= \lim_{x \rightarrow 1} \frac{(x+1-2)(\sqrt{x+3} + 2)}{(x+3-4)(\sqrt{x+1} + \sqrt{2})} = \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x+3} + 2)}{(x-1)(\sqrt{x+1} + \sqrt{2})} = \lim_{x \rightarrow 1} \frac{\sqrt{x+3} + 2}{\sqrt{x+1} + \sqrt{2}} = \\ &= \frac{\sqrt{1+3} + 2}{\sqrt{1+1} + \sqrt{2}} = \frac{2+2}{\sqrt{2} + \sqrt{2}} = \frac{4}{2\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}. \end{aligned}$$

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 5. Remarkable Limits

The purpose of the lesson is to learn how to calculate the limits of functions using the first and second remarkable limits, their consequences, as well as the wonderful limits associated with exponential and logarithmic functions.

Issues for discussion:

1. The first remarkable limit, the consequences.
2. The second remarkable limit, the consequences.
3. Remarkable limits associated with exponential and logarithmic functions.

Tasks solved in the classroom

Example 20. Calculate the limit of the function $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} &= \left(\frac{0}{0} \right) = \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x^2} = \\ &= 2 \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2} \cdot 2} \right)^2 = 2 \lim_{x \rightarrow 0} \frac{1}{4} \frac{\sin \frac{x}{2}}{\frac{x}{2}} = \frac{1}{2}. \end{aligned}$$

Example 21. Calculate the limit of the function $\lim_{x \rightarrow 0} \frac{\operatorname{tg} 5x}{\sin 7x}$.

$$\lim_{x \rightarrow 0} \frac{\operatorname{tg} 5x}{\sin 7x} = \left(\frac{0}{0} \right) = \frac{5}{7} \lim_{x \rightarrow 0} \frac{\operatorname{tg} 5x}{5x} \cdot \lim_{x \rightarrow 0} \frac{5x}{\sin 5x} = \frac{5}{7}.$$

Example 22. Calculate the limit of the function $\lim_{x \rightarrow \pi/4} \frac{\sin x - \cos x}{\pi - 4x}$.

$$\lim_{x \rightarrow \pi/4} \frac{\sin x - \cos x}{\pi - 4x} = \left(\frac{0}{0} \right) = \lim_{x \rightarrow \pi/4} \frac{-\frac{2}{\sqrt{2}} \sin(\pi/4 - x)}{\pi - 4x} = \lim_{x \rightarrow \pi/4} \frac{-\sin(\pi/4 - x)}{2\sqrt{2}(\pi/4 - x)} = -\frac{1}{2\sqrt{2}}.$$

Example 23. Calculate the limit of the function $\lim_{x \rightarrow \pi/2} \frac{\cos x}{\pi - 2x}$.

Let's calculate the limit using the substitution of the variable $y = \pi/2 - x$, from where
 $x = \pi/2 - y$ and $\pi - 2x = \pi - \pi + 2y$, then

$$\lim_{x \rightarrow \pi/2} \frac{\cos x}{\pi - 2x} = \left(\frac{0}{0} \right) = \lim_{y \rightarrow 0} \frac{\cos(\pi/2 - y)}{2y} = \lim_{y \rightarrow 0} \frac{\sin y}{y} = \frac{1}{2}.$$

Example 24. Calculate the limit of the function $\lim_{x \rightarrow \infty} \left(\frac{x+3}{x-1} \right)^{x+3}$.

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\frac{x+3}{x-1} \right)^{x+3} &= (1^\infty) = \lim_{x \rightarrow \infty} \left(\frac{x-1+4}{x-1} \right)^{x+3} = \lim_{x \rightarrow \infty} \left(1 + \frac{4}{x-1} \right)^{x+3} = \\ &= \lim_{x \rightarrow \infty} \left(1 + \frac{4}{x-1} \right)^{\frac{x-1}{4} \cdot \frac{4}{x-1} \cdot (x+3)} = e^{\lim_{x \rightarrow \infty} \left(\frac{4}{x-1} \right) \cdot (x+3)} = e^4 \end{aligned}$$

Tasks for independent work

Calculate the limit of the function:

- | | |
|---|--|
| a). $\lim_{x \rightarrow 2} \frac{x^2 - 5x + 6}{x^3 - 2x^2 - x + 2};$ | b). $\lim_{x \rightarrow -1} \frac{x^3 - 3x - 2}{x^2 - 1};$ |
| c). $\lim_{x \rightarrow -1} \frac{x^3 + 2x^2 + 5x + 4}{x^3 - x^2 - 3x - 1};$ | d). $\lim_{x \rightarrow -2} \frac{x^3 + 5x^2 + 8x + 4}{x^3 + 3x^2 - 4};$ |
| e). $\lim_{x \rightarrow 1} \frac{\sqrt{5-x} - 2}{\sqrt{2-x} - 1};$ | f). $\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 1} - 1}{\sqrt{x^2 + 16} - 4};$ |
| g). $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1+x^2}}{1 - \sqrt{1+x}};$ | h). $\lim_{x \rightarrow 0} \frac{\sin 3x}{6x};$ |
| i). $\lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 10x};$ | j). $\lim_{x \rightarrow 0} \frac{\operatorname{tg} 5x}{15x};$ |
| k). $\lim_{x \rightarrow 0} \frac{\operatorname{tg} 7x}{\operatorname{tg} 3x};$ | m). $\lim_{x \rightarrow 0} \frac{1}{\operatorname{ctg} 4x \sin 5x};$ |
| m). $\lim_{x \rightarrow 0} \frac{1 - \cos 4x}{6x^2};$ | a). $\lim_{x \rightarrow 0} \frac{1 - \cos 6x}{\sin^2 4x};$ |
| p). $\lim_{x \rightarrow 0} \frac{\operatorname{tg}^2 3x \sin 5x}{(1 - \cos 4x) \operatorname{tg} 2x};$ | p). $\lim_{x \rightarrow 0} \frac{\operatorname{tg}^2 3x \sin^2 5x}{x^3 \sin 4x};$ |

$$\text{c). } \lim_{x \rightarrow 0} \frac{\sin 7x - \sin 2x}{\sin x};$$

$$\text{t). } \lim_{x \rightarrow 0} \frac{\cos x - \cos 5x}{\cos 2x - \cos 8x};$$

$$\text{y). } \lim_{x \rightarrow 0} \frac{\sin 5x}{\sqrt{x+4} - 2};$$

$$\text{f). } \lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - 1}{\sin 4x}.$$

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 6. Continuity of the function

The purpose of the lesson: to learn how to study functions for continuity, to determine the classification of points of discontinuity of the function.

Issues for discussion:

1. The concept of continuity of a function.
2. Function break points and their classification.
3. Properties of functions that are continuous at a point.
4. Properties of functions that are continuous on a segment.

Tasks solved in the classroom

Example 25. A function is given. Find its break points, if any, and the function jump at each break point:

$$1) f(x) = \frac{1}{x^2 - 4};$$

$$2) f(x) = \frac{3x - 5}{x^2 + 2x + 10};$$

$$3) f(x) = \frac{x - 8}{x^2 - 11x + 24};$$

$$4) f(x) = \operatorname{arccotg} \frac{1}{x}.$$

1) The zeros of the denominator of the function $f(x) = \frac{1}{x^2 - 4}$ are determined from the equation $x^2 - 4 = 0$, from which the possible break points follow $x = \pm 2$. To check whether these points are really break points, let's calculate the one-way limits:

$$\lim_{x \rightarrow -2-0} \frac{1}{x^2 - 4} = +\infty, \quad \lim_{x \rightarrow -2+0} \frac{1}{x^2 - 4} = -\infty, \quad \lim_{x \rightarrow 2-0} \frac{1}{x^2 - 4} = -\infty, \quad \lim_{x \rightarrow 2+0} \frac{1}{x^2 - 4} = +\infty.$$

As a result, it is easy to see that the points $x = -2$ and $x = 2$ are the points of the second kind of rupture.

2) Consider the function $f(x) = \frac{3x - 5}{x^2 + 2x + 10}$. Let us find the zeros of the denominator

from the equation $x^2 + 2x + 10 = 0$. It is not difficult to see that it has no real roots. Thus, the function $f(x)$ is continuous over the entire number line.

3) The breaking points in this case can be the zeros of the denominator, i.e. $x = 8$ and $x = 3$. At the point $x = 8$, the function is not defined, since when substituting $x = 8$ in a fraction we have uncertainty $\left(\frac{0}{0}\right)$, let us calculate the limit of the function at $x \rightarrow 8$:

$$\lim_{x \rightarrow 8} \frac{x-8}{x^2-11x+24} = \left(\frac{0}{0}\right) = \lim_{x \rightarrow 8} \frac{x-8}{x^2-11x+24} = \lim_{x \rightarrow 8} \frac{x-8}{(x-8)(x-3)} = \lim_{x \rightarrow 8} \frac{1}{x-3} = \frac{1}{5},$$

Therefore, the point $x=8$ is the point of the avoidable gap. Let's examine the point $x=3$, for this we will calculate the one-way limits:

$$\lim_{x \rightarrow 3-0} \frac{x-8}{x^2-11x+24} = \lim_{x \rightarrow 3-0} \frac{1}{x-3} = -\infty, \quad \lim_{x \rightarrow 3+0} \frac{x-8}{x^2-11x+24} = \lim_{x \rightarrow 3+0} \frac{1}{x-3} = +\infty,$$

On the basis of which we conclude that the point $x=3$ is a point of infinite rupture (of the second kind).

4) Let us first consider the function $f(y) = \text{arctg } y$. Since the domain of this function is the entire numerical line, and it nowhere goes to infinity, the function $f(y)$ is continuous over the entire domain of definition. Therefore, the only possible point of break of the function $f(x) = \text{arctg } \frac{1}{x}$ can be the point $x=0$, in order to check whether the function $f(x)$ is really broken here, let's calculate the one-way limits:

$$\lim_{x \rightarrow 0-0} \text{arctg } \frac{1}{x} = \text{arctg}(-\infty) = \pi, \quad \lim_{x \rightarrow 0+0} \text{arctg } \frac{1}{x} = \text{arctg}(+\infty) = 0.$$

As a result, we have that the function $x=0$ $f(x) = \text{arctg } \frac{1}{x}$ suffers a discontinuity of the first kind, making a jump $d = f(+0) - f(-0) = -\pi$.

Example 26. Examine the continuity of the function and determine the type of break points, if any

$$f(x) = \begin{cases} x+4, & x < -1 \\ x^2+2, & -1 \leq x \leq 1 \\ 2x, & x \geq 1 \end{cases}$$

Let's find one-sided limits:

$$\lim_{x \rightarrow -1-0} f(x) = 3$$

$$\lim_{x \rightarrow -1+0} f(x) = 3$$

$$\lim_{x \rightarrow 1-0} f(x) = 3$$

$$\lim_{x \rightarrow 1+0} f(x) = 2$$

Thus, at the point $x=-1$ the function is continuous, the point $x=1$ is the point of discontinuity of the first kind, let us determine the jump of the function at this point $d = f(x_0+0) - f(x_0-0) = 2 - 3 = -1$ (Fig. 23).

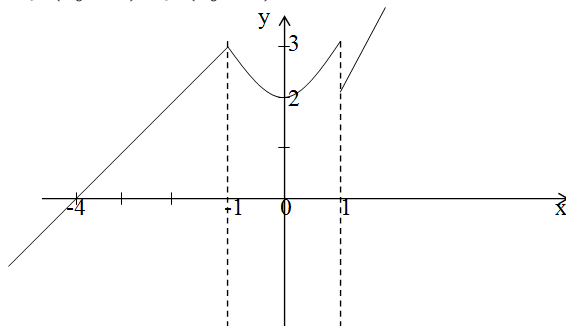


Fig. 23

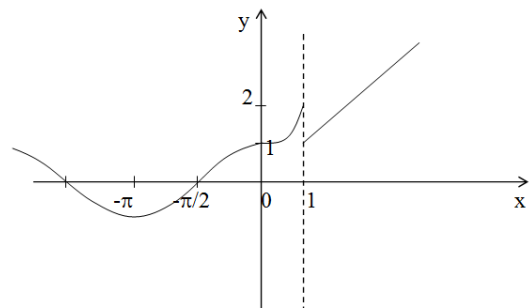


Fig. 24

Example 27. Examine the continuity of the function and determine the type of break points, if any

$$f(x) = \begin{cases} \cos x, & x \leq 0 \\ x^2 + 1, & 0 < x < 1. \\ x, & x \geq 1 \end{cases}$$

Let's find one-sided limits:

$$\begin{array}{ll} \lim_{x \rightarrow 0-0} f(x) = 1 & \lim_{x \rightarrow 1-0} f(x) = 2 \\ \lim_{x \rightarrow 0+0} f(x) = 1 & \lim_{x \rightarrow 1+0} f(x) = 1 \end{array}$$

We come to the conclusion that at point $x = 0$ the function is continuous, *point* $x = 1$ is a point of discontinuity of the first kind, and we will determine the jump of the function at this point $d = f(x_0 + 0) - f(x_0 - 0) = 1 - 2 = -1$ (Fig. 24).

Tasks for independent work

Investigate these functions for continuity

$$\text{a). } f(x) = 2^{\frac{1}{x-3}} + 1; \quad \text{b). } f(x) = \frac{x-5}{x+3};$$

$$\text{c). } f(x) = 4^{\frac{1}{3-x}} + 2; \quad \text{d). } f(x) = 9^{\frac{1}{2-x}}.$$

3. To study these functions for continuity, to construct graphs of functions

$$\begin{array}{l} \text{a). } f(x) = \begin{cases} x+4, & x < -1, \\ x^2 + 2, & -1 \leq x < 1, \\ 2x, & x \geq 1. \end{cases} \\ \text{b). } f(x) = \begin{cases} -x, & x \leq 0, \\ -(x-1)^2, & 0 < x < 2, \\ x-3, & x \geq 2. \end{cases} \\ \text{c). } f(x) = \begin{cases} \cos x, & x \leq \pi/2 \\ 0, & \pi/2 < x < \pi, \\ 2, & x \geq \pi. \end{cases} \\ \text{d). } f(x) = \begin{cases} \sqrt{1-x}, & x \leq 0, \\ 0, & 0 < x \leq 2, \\ x-2, & x > 2. \end{cases} \end{array}$$

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

SECTION II. DIFFERENTIAL CALCULUS FOR FUNCTIONS OF A SINGLE VARIABLE

Lesson No 7. Calculating a derivative by definition

The purpose of the lesson: to define a derived function as the limit of a difference relation, to master the technique of finding such limits for elementary functions.

Issues for discussion:

1. Definition of a derivative, its geometric and mechanical meaning.
2. Continuity of the differentiated function.

Tasks solved in the classroom

Example 28. Using the definition of a derived function (a method of direct differentiation), find the derivative for $y = \frac{2}{x+3}$.

1. Find the incremental value of the function: $y + \Delta y = \frac{2}{x + \Delta x + 3}$.

2. Find the increment of the function Δy :

$$\Delta y = \frac{2}{x + \Delta x + 3} - \frac{2}{x + 3} = \frac{2x + 6 - 2x - 2 \cdot \Delta x - 6}{(x + \Delta x + 3) \cdot (x + 3)} = \frac{-2 \cdot \Delta x}{(x + \Delta x + 3) \cdot (x + 3)}.$$

3. Make a difference relation $\frac{\Delta y}{\Delta x}$: $\frac{\Delta y}{\Delta x} = \frac{-2}{(x + \Delta x + 3) \cdot (x + 3)}$.

4. Find the limit of the ratio $\frac{\Delta y}{\Delta x}$ when $\Delta x \rightarrow 0$:

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-2}{(x + \Delta x + 3) \cdot (x + 3)} = -\frac{2}{(x + 3)^2}.$$

Therefore, the derivative $y' = -\frac{2}{(x + 3)^2}$ of .

Example 29. Using the differentiation formulas, find the outputs of the following functions

1) $y = x^2 - 5x + 4$;

2) $y = \sqrt{x} + \frac{5}{\sqrt[3]{x}} - \frac{1}{x^2} + \frac{1}{3x^3}$;

3) $z = x^5 \left(2 - \frac{x}{3} + 3x^2 \right)$;

4) $f(x) = \frac{x^2}{x^2 + 1}$.

1) $y' = (x^2 - 5x + 4)' = (x^2)' - (5x)' + (4)' = 2x - 5$.

2) Transform this function using fractional and negative exponents

$$y = \sqrt{x} + \frac{5}{\sqrt[3]{x}} - \frac{1}{x^2} + \frac{1}{3x^3} = x^{1/2} + 5x^{-1/3} - x^{-2} + \frac{1}{3}x^{-3}.$$

$$y' = \left(x^{1/2} + 5x^{-1/3} - x^{-2} + \frac{1}{3}x^{-3} \right)' = \frac{1}{2}x^{-1/2} - \frac{5}{3}x^{-4/3} + 2x^{-3} + \frac{1}{3}(-3)x^{-4} = \frac{1}{2\sqrt{x}} - \frac{5}{3\sqrt[3]{x^4}} + \frac{2}{x^3} - \frac{1}{x^4}$$

3) 1st method. Let's use the formula of the derivative of the product

$$z' = (x^5)' \left(2 - \frac{x}{3} + 3x^2 \right) + x^5 \left(2 - \frac{x}{3} + 3x^2 \right)' = 5x^4 \left(2 - \frac{x}{3} + 3x^2 \right) + x^5 \left(-\frac{1}{3} + 6x \right) = 10x^4 - \frac{5}{3}x^5 + 15x^6 - \frac{1}{3}x^5 + 6x^6 = 21x^6 - 2x^5 + 10x^4$$

Method 2. First, we transform the original expression, and then we perform differentiation

$$z = 2x^5 - \frac{1}{3}x^6 + 3x^7, \quad z' = 10x^4 - 2x^5 + 21x^6.$$

It is easy to see that in a number of tasks this method leads to results faster.

4) Calculate the derivative as a quotient derivative

$$f'(x) = \left(\frac{x^2}{x^2 + 1} \right)' = \frac{(x^2)'(x^2 + 1) - (x^2 + 1)'x^2}{(x^2 + 1)^2} = \frac{2x(x^2 + 1) - 2x \cdot x^2}{(x^2 + 1)^2} = \frac{2x^3 + 2x - 2x^3}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}.$$

Tasks for independent work

1. Find derivatives of these functions by direct differentiation

a). $y = x^3 - 2x + 3$; b). $y = \frac{x+1}{x-3}$; ;

c). $y = \sqrt{2x-3}$; d). $y = \ln x$. .

2. Using differentiation formulas, find derivatives of the following functions

a). $y = x + 3x^2 - \frac{x^2}{3}$;

g). $y = -3\cos t \operatorname{ctg} t$;

b). $y = x - 2\sqrt{x}$;

h). $r = \frac{1 + \cos \varphi}{\sin \varphi}$;

c). $s = \frac{1}{t} + \frac{1}{t^2}$;

i). $f(x) = \frac{\sqrt{x}}{1 + \sqrt{x}}$; calculate $f'(1)$;

d). $z = 3\sqrt[3]{x} - 2\sqrt{x^3} + 4$;

j). $r(\varphi) = \varphi \sin \varphi + \cos \varphi$; calculate $r'(\pi)$;

e). $u = \frac{2t}{t+3}$;

k). $u(r) = \frac{r^2}{2x^2} - \frac{2x^2}{r^2}$; calculate $\left(\frac{du}{dr} \right)_{r=x}$;

f). $v = \frac{x^2 - 3}{x^2 + 3}$;

m). $z = (y^2 - 2y) \operatorname{tg} y$; calculate $z'(0)$.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 8. Technique for finding derivatives

The purpose of the lesson: to give a definition of a derived function, to master the technique of finding derivatives of elementary and complex functions, using the rules of differentiation and the table of derivatives.

Issues for discussion:

1. Rules of differentiation. Table of derivatives.
2. A derivative of a complex function.

Tasks solved in the classroom

Example 30. Find a derivative for each function

$$1). y = x \cos x \sin x + \frac{1}{2} \cos^2 x.$$

First, let's perform the transformations $y = \frac{x}{2} \sin 2x + \frac{1}{2} \cos^2 x$,

$$\begin{aligned} y' &= \frac{1}{2} \sin 2x + \frac{1}{2} x 2 \cos 2x + \frac{1}{2} 2 \cos x (-\sin x) = \\ &= \frac{1}{2} \sin 2x + x \cos 2x - \sin x \cos x = x \cos 2x. \end{aligned}$$

$$2). y = \frac{x^2 e^{x^2}}{x^2 + 1}.$$

$$\begin{aligned} y' &= \frac{(2xe^{x^2} + x^2 2xe^{x^2})(x^2 + 1) - (2x)x^2 e^{x^2}}{(x^2 + 1)^2} = \\ &= \frac{2x^3 e^{x^2} + 2x^5 e^{x^2} + 2xe^{x^2} + 2x^3 e^{x^2} - 2x^3 e^{x^2}}{(x^2 + 1)^2} = \frac{2xe^{x^2}(x^4 + 1 + x^2)}{(x^2 + 1)^2}. \end{aligned}$$

$$3). y = \ln \operatorname{tg} \frac{x}{2} - \frac{x}{\sin x}.$$

$$\begin{aligned} y' &= \frac{1}{\operatorname{tg} \frac{x}{2}} \cdot \frac{1}{\cos^2 \frac{x}{2}} \cdot \frac{1}{2} - \frac{\sin x - x \cos x}{\sin^2 x} = \frac{1}{2 \sin \frac{x}{2} \cos \frac{x}{2}} - \frac{\sin x - x \cos x}{\sin^2 x} = \\ &= \frac{\sin x - \sin x + x \cos x}{\sin^2 x} = \frac{x \cos x}{\sin^2 x}. \end{aligned}$$

$$4). y = \operatorname{arctg} \frac{2x^4}{1 - x^8}.$$

$$y' = \frac{1}{\left(1 + \frac{4x^8}{(1-x^8)^2}\right)} \cdot \frac{8x^3(1-x^8) - (-8x^7)2x^4}{(1-x^8)^2} = \frac{(1-x^8)^2(8x^3-8x^{11}+16x^{11})}{(1+x^8)^2(1-x^8)^2} =$$

$$= \frac{8x^3+8x^{11}}{(1+x^8)^2} = \frac{8x^3(1+x^8)}{(1+x^8)^2} = \frac{8x^3}{1+x^8}.$$

5). $y = x^2 e^{x^2} \ln x.$

$$y' = \left(x^2 e^{x^2}\right)' \ln x + x^2 e^{x^2} \frac{1}{x} = \left(2xe^{x^2} + x^2 e^{x^2} 2x\right) \ln x + xe^{x^2} =$$

$$= 2xe^{x^2} (1+x^2) \ln x + xe^{x^2} = xe^{x^2} (1+2\ln x + 2x^2 \ln x).$$

Tasks for independent work

Find derivatives of specified functions

a). $y = \sin(x^3 + 3x);$

b). $y = (2x^4 - 5x + 1)^3;$

c). $y = 5^{\cos x};$

d). $y = \ln(1 + \sin x);$

e). $y = \arcsin \sqrt{x};$

f). $y = e^{\operatorname{tg} x} \cdot \cos^3 x;$

g). $y = \ln \sin 5x;$

h). $y = e^{\cos x^2};$

i). $y = e^{\arcsin \frac{1}{x}};$

j). $y = \arccos \sqrt{1-x^2};$

k). $y = \ln(x + \sqrt{x^2 + a^2});$

m). $y = (2^{\operatorname{arctg} \sqrt{x}} + 1)^3;$

n). $y = \arcsin \frac{x}{\sqrt{1+x^2}};$

a). $y = \operatorname{arctg} \sqrt{x^2 - 1} - \frac{\ln x}{\sqrt{x^2 - 1}};$

p). $y = \ln(e^x + \sqrt{1 + e^{2x}}),$ calculate $y'(0).$

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 9. Differentiation of complex functions

The purpose of the lesson: to master the technique of finding derivatives of complex functions, using the rules of differentiation and the table of derivatives.

Issues for discussion:

1. A derivative of a complex and inverse function.
2. Differentiation of implicit functions. Logarithmic differentiation.
3. Differential of the function, its geometric meaning. Properties of the differential.
4. Derivatives and differentials of higher orders. Leibniz's formula.

Tasks solved in the classroom

Example 31. Find the derivative of the y' function y defined by the equation $x^3 + x^2y + y^2 = 0$.

We differentiate both parts of the equation by the variable x , assuming that y there is a function of x : $3x^2 + 2xy + x^2y' + 2yy' = 0$, we resolve the resulting equation with respect to the desired derivative y' : $y'(x^2 + 2y) = -3x^2 - 2xy$, $y' = -\frac{3x^2 + 2xy}{x^2 + 2y}$.

Example 32. Using the method of logarithmic differentiation, find the derivative of the function $y = x \cdot \sqrt[3]{\frac{x^2}{x^2 + 1}}$.

Logarithming both parts of the equation, we get

$$\ln y = \ln x + \frac{1}{3} \cdot \ln x^2 - \frac{1}{3} \cdot \ln(x^2 + 1) = \ln x + \frac{2}{3} \cdot \ln x - \frac{1}{3} \cdot \ln(x^2 + 1),$$

$$\text{or } \ln y = \frac{5}{3} \cdot \ln x - \frac{1}{3} \cdot \ln(x^2 + 1).$$

Differentiating both parts of the obtained equality, we have

$$\frac{1}{y} \cdot y' = \frac{5}{3} \cdot \frac{1}{x} - \frac{1}{3} \cdot \frac{2x}{x^2 + 1}; \quad \frac{1}{y} \cdot y' = \frac{5x^2 + 5 - 2x^2}{3x \cdot (x^2 + 1)};$$

$$\frac{1}{y} \cdot y' = \frac{3x^2 + 5}{3x \cdot (x^2 + 1)},$$

$$\text{whence it follows that } y' = y \cdot \frac{3x^2 + 5}{3x \cdot (x^2 + 1)} \Rightarrow y' = x \cdot \sqrt[3]{\frac{x^2}{x^2 + 1}} \cdot \frac{3x^2 + 5}{3x \cdot (x^2 + 1)}.$$

Tasks for independent work

1. Find the derivative of y' the function y given in the implicit form

a). $x^3 + x^2y + y^2 = 0$;

b). $\operatorname{tg} y - xy = 0$;

c). $\cos(x - y) - y = 0$;

d). $x^2y + \operatorname{arctg} \frac{x}{y} = 0$;

e). $e^{x+y} - \ln \sin \frac{y}{x} = 0$

2. Using the method of logarithmic differentiation, find derivatives of these functions

$$\begin{array}{ll} \text{a) } y = \frac{x \cdot (1+x^2)^2}{\sqrt{1-x^2}}; & \text{b) } y = x^3 \cdot \sqrt{\frac{1+x^2}{1+x}}; \\ \text{c) } y = \sqrt[3]{\frac{x \cdot (x^3+1)}{\cos^2 x}}; & \text{d) } y = \frac{e^{\sin^2 x} \cdot \sqrt[3]{\arcsin x}}{(1+x)^3}. \end{array}$$

3. Using the method of logarithmic differentiation, find derivatives of exponentially power functions

$$\begin{array}{lll} \text{a) } y = x^x; & \text{b) } y = (\cos x)^{\sin x}; & \text{c) } y = (\operatorname{tg} x)^{\operatorname{ctg} x}; \\ \text{d) } y = (1-x^2)^{\arccos x}; & \text{e) } y = (\arctg x)^{\arcsin^2 x}. \end{array}$$

Tasks solved in the classroom

Example 33. Find the differential dy of the function $y = e^{x^2+1}$.

Based on the definition of the differential, we have

$$dy = y' \cdot dx = (e^{x^2+1})' \cdot dx = e^{x^2+1} \cdot 2x \cdot dx.$$

This function can be represented as follows: $y = e^u$, where $u = x^2 + 1$, then we will have $dy = e^u \cdot du = e^{x^2+1} \cdot d(x^2 + 1) = e^{x^2+1} \cdot 2x \cdot dx$.

Example 34. Find the Differential of the Dy Function $y = \arctg e^{\sin 3x}$.

Find the derivative y' , using the rule of differentiation of a complex function:

$$y' = \frac{1}{1 + (e^{\sin 3x})^2} \cdot e^{\sin 3x} \cdot \cos 3x \cdot 3, \text{ then}$$

$$dy = \frac{1}{1 + (e^{\sin 3x})^2} \cdot e^{\sin 3x} \cdot \cos 3x \cdot 3 \cdot dx.$$

The dy differential can also be found directly

$$\begin{aligned} dy &= d(\arctg e^{\sin 3x}) = \frac{1}{1 + (e^{\sin 3x})^2} \cdot d(e^{\sin 3x}) = \\ &= \frac{1}{1 + (e^{\sin 3x})^2} \cdot e^{\sin 3x} \cdot d(\sin 3x) = \frac{1}{1 + (e^{\sin 3x})^2} \cdot e^{\sin 3x} \cdot \cos 3x \cdot 3 \cdot dx. \end{aligned}$$

Example 35. Find the differential dy of a function y given implicitly by the equation $x \cdot \sin y + y^2 \cdot \cos x - 1 = 0$.

Find the derivative y' , to do this, differentiate both sides of the equation by x , considering y to be a function of the argument x

$$\sin y + x \cdot \cos y \cdot y' + 2y \cdot y' \cdot \cos x - y^2 \cdot \sin x = 0.$$

From the resulting equality, we will express y'

$$y' \cdot (x \cdot \cos y + 2y \cdot \cos x) = y^2 \cdot \sin x - \sin y; \quad y' = \frac{y^2 \cdot \sin x - \sin y}{x \cdot \cos y + 2y \cdot \cos x}.$$

$$\text{Therefore, the differential } dy = \frac{y^2 \cdot \sin x - \sin y}{x \cdot \cos y + 2y \cdot \cos x} \cdot dx.$$

Example 36. Find the approximate value of the function $f(x) = \sqrt{x^2 + 7}$ at $x=3.02$ based on its exact value at $x=3$.

Let $x_1=3$ and $x_2=3,02$. Applying the approximate equation (5.7), we get:
 $f(3,02) = \sqrt{(3,02)^2 + 7} \approx \sqrt{3^2 + 7} + f'(3) \cdot (3,02 - 3) \approx 4 + f'(3) \cdot 0,02$.

To find $f'(3)$, you must first differentiate this function and then find the numerical value of the derivative at $x=3$.

$$f'(x) = \frac{2x}{2\sqrt{x^2 + 7}} = \frac{x}{\sqrt{x^2 + 7}}, \quad f'(3) = \frac{3}{4}.$$

$$\text{Thus, } \sqrt{(3,02)^2 + 7} \approx 4 + \frac{3}{4} \cdot 0,02 = 4,015.$$

Example 37. Find the approximate value of the value of . $\operatorname{tg} 47^\circ$

Let us consider the function $y = \operatorname{tg} x$. It is known that $\operatorname{tg} 45^\circ = 1$. It is therefore convenient to put $x_1 = 45^\circ$ and $x_2 = 47^\circ$. In order to take advantage of the approximate equation (5.7), it is necessary to first find the value $y = \operatorname{tg} x$ of the function and its derivative $y' = \frac{1}{\cos^2 x}$ at

$$x_1 = 45^\circ = \frac{\pi}{4}: y\left(\frac{\pi}{4}\right) = 1, \quad y'\left(\frac{\pi}{4}\right) = 2. \text{ Let us express the difference in radian measure}$$

$$x_2 - x_1 = \frac{\pi}{180} \cdot 47 - \frac{\pi}{4} = \frac{47\pi - 45\pi}{180} = \frac{\pi}{90} = 0,035.$$

$$\text{Therefore, } \operatorname{tg} 47^\circ \approx 1 + 2 \cdot 0,035 = 1,070.$$

Example 38. Find the approximate value of the value of . $A = \sqrt[3]{26,46}$

Consider the function . Since $y = \sqrt[3]{x}$ the function $y = 3$ at $x=27$, it is convenient to put $x_1=27$ and $x_2=26.46$. Let's calculate the value of the derivative $y' = \frac{1}{3 \cdot \sqrt[3]{x^2}}$ at $x_1=27$:

$$y'(27) = \frac{1}{3 \cdot 3^2} = \frac{1}{27}. \text{ Using the approximate equation (5.7), we have}$$

$$A = \sqrt[3]{26,46} \approx \sqrt[3]{27} + \frac{1}{27} \cdot (26,46 - 27) = 3 - 0,02 = 2,98.$$

Tasks for independent work

1. Find Feature Differentials

$$\begin{array}{ll} \text{a). } y = \sin 5x - 5 \cdot e^{-x^2}; & \text{b). } y = \ln \frac{1-x}{1+x}; \\ \text{c). } y = \ln \operatorname{tg} \left(\frac{\pi}{2} - \frac{x}{4} \right); & \text{d). } y = 5^{\sqrt{\operatorname{arctg}(x^2)}}. \end{array}$$

2. Using the approximate equation (5.7), calculate the approximate value of the specified functions for the specified values of the argument

$$\text{a). } y = \sqrt{x^3 - 2}, x=3.02; \text{ b). } y = \sqrt{\frac{2-x}{2+x}}, x=0.02; \text{ c). } y = e^{1-x^2}, x=1.05.$$

3. Using the approximate equation (5.7), calculate the approximate value of the

specified values

a). $\cos 61^\circ$; b). $\operatorname{tg} 44^\circ$; c). $e^{0.2}$; d). $\sqrt[3]{7.76}$; e). $\operatorname{arctg} 1.05$; f). $\arcsin 0.54$.

Tasks solved in the classroom

Example 39. For these functions, find derivatives of the specified order

$$1) y = x^5 - 7x^3 + 2; y''' = ? \quad 2) y = \ln x; y^{(5)} = ?$$

$$3) s = \operatorname{arctg} 2x; s'''(-1) = ?$$

1) Let us perform a sequential differentiation of the function y

$$y' = (x^5 - 7x^3 + 2)' = 5x^4 - 21x^2,$$

$$y'' = (5x^4 - 21x^2)' = 20x^3 - 42x,$$

$$y''' = (20x^3 - 42x)' = 60x^2 - 42;$$

2) Performing a sequential differentiation of a function $y = \ln x$, we have

$$y' = (\ln x)' = \frac{1}{x}, \quad y'' = \left(\frac{1}{x}\right)' = -\frac{1}{x^2}, \quad y''' = \left(-\frac{1}{x^2}\right)' = \frac{2}{x^3},$$

$$y^{(4)} = \left(\frac{2}{x^3}\right)' = -\frac{6}{x^4}, \quad y^{(5)} = \left(-\frac{6}{x^4}\right)' = \frac{24}{x^5};$$

3) First, let's perform a sequential differentiation of the function $s = \operatorname{arctg} 2x$:

$$s' = (\operatorname{arctg} 2x)' = \frac{1}{1 + (2x)^2} \cdot 2 = \frac{2}{1 + 4x^2},$$

$$s'' = \left(\frac{2}{1 + 4x^2}\right)' = -\frac{2}{(1 + 4x^2)^2} (1 + 4x^2)' = -\frac{16x}{(1 + 4x^2)^2},$$

$$\begin{aligned} s''' &= \left(-\frac{16x}{(1 + 4x^2)^2}\right)' = -\frac{(16x)'(1 + 4x^2)^2 - 16x((1 + 4x^2)^2)'}{(1 + 4x^2)^4} = \\ &= -\frac{16(1 + 4x^2)^2 - 16x \cdot 2(1 + 4x^2) \cdot 8x}{(1 + 4x^2)^4} = -\frac{16 + 64x^2 - 256x^2}{(1 + 4x^2)^3} = \frac{192x^2 - 16}{(1 + 4x^2)^3}. \end{aligned}$$

Substituting the value of the argument $x = -1$, we have

$$s'''(-1) = \frac{192 \cdot (-1)^2 - 16}{(1 + 4(-1)^2)^3} = \frac{192 - 16}{(1 + 4)^3} = \frac{176}{125}.$$

Example 40. Find the fourth-order derivative of the function $y = x^3 \cdot \sin x$.

Assuming in Leibniz's formula (5.12) $n = 4$, we have
 $(u \cdot v)^{(IV)} = u^{(IV)}v + 4u'''v' + 6u''v'' + 4u'v''' + uv^{(IV)}$. Let $u = x^3$, $v = \sin x$,

$$u = x^3, u' = 3x^2, u'' = 6x, u''' = 6, u^{IV} = 0;$$

$$v = \sin x, v' = \cos x, v'' = -\sin x, v''' = -\cos x, v^{IV} = \sin x,$$

then

$$\begin{aligned} (x^3 \cdot \sin x)' &= 4 \cdot 6 \cdot \cos x + 6 \cdot 6x \cdot (-\sin x) + 4 \cdot 3x^2 \cdot (-\cos x) + x^3 \cdot \sin x = \\ &= 24 \cos x - 36x \sin x - 12x^2 \cos x + x^3 \sin x. \end{aligned}$$

Example 41. Find a common expression of n the derived order of the function $y = \ln x$

Let's find the first few derivatives

$$\begin{aligned} y' &= (\ln x)' = \frac{1}{x}, & y'' &= \left(\frac{1}{x}\right)' = -\frac{1}{x^2}, & y''' &= \left(-\frac{1}{x^2}\right)' = \frac{2}{x^3}, \\ y^{(4)} &= \left(\frac{2}{x^3}\right)' = -\frac{2 \cdot 3}{x^4}, & y^{(5)} &= \left(-\frac{6}{x^4}\right)' = \frac{2 \cdot 3 \cdot 4}{x^5}, \end{aligned}$$

whence the derivative of n the n -th order of the function becomes apparent $y = \ln x$

$$y^{(n)} = (-1)^{n+1} \frac{(n-1)!}{x^n}.$$

Example 42. Find the second-order derivative of the function y implicitly given by the equation $x^2 + 2x + y^2 - 5 = 0$.

Differentiating both parts of the equation by the variable x and considering y to be a function of x , we have $2x + 2 + 2y \cdot y' = 0$, from which $y' = -\frac{x+1}{y}$ (1). Differentiating both parts of the equation (1) by the variable x , while considering y as a function of x , we get $y'' = -\frac{y - (x+1) \cdot y'}{y^2}$ (2). Replacing it in the right part (2) y' with the expression of equation

$$(1), \text{ we get } y'' = -\frac{y + \frac{(x+1)^2}{y}}{y^2} = -\frac{y^2 + (x+1)^2}{y^3}.$$

Example 43. Find y''' if the function is given by the equation $e^y + y - x = 0$.

$$\begin{aligned} e^y \cdot y' + y' - 1 &= 0, & y' &= \frac{1}{e^y + 1}, \\ y'' &= -\frac{e^y \cdot y'}{(e^y + 1)^2}, & y'' &= -\frac{e^y}{(e^y + 1)^3}, \\ y''' &= -\frac{e^y \cdot y' \cdot (e^y + 1)^3 - e^y \cdot 3 \cdot (e^y + 1)^2 \cdot e^y \cdot y'}{(e^y + 1)^6} = \end{aligned}$$

$$= -\frac{e^y \cdot y' \cdot (e^y + 1 - 3 \cdot e^y)}{(e^y + 1)^4} = -\frac{e^y \cdot (1 - 2e^y)}{(e^y + 1)^5}.$$

Example 44. Find a second-order derivative $\frac{d^2 y}{dx^2}$ if the function y of the argument x is given by parametric equations $\begin{cases} x = r \cdot \cos t, \\ y = r \cdot \sin t \end{cases}$.

Find the first-order derivative $\frac{dy}{dx} = y'_x$. Since $\frac{dx}{dt} = -r \cdot \sin t$, a, $\frac{dy}{dt} = r \cdot \cos t$ then

$y'_x = \frac{dy}{dx} = -ctg t$. To find the second-order derivative $y'' = \frac{d^2 y}{dx^2} = \frac{d(y'_x)}{dx}$, consider the first-order derivative as a y'_x function of the parameter t and again apply the formula (5.12). Since

$$\frac{d(y'_x)}{dt} = \frac{d(-ctg t)}{dt} = \frac{1}{\sin^2 t}, \text{ a } \frac{dx}{dt} = -r \cdot \sin t,$$

then

$$\frac{d^2 y}{dx^2} = \frac{1}{\sin^2 t} \div (-r \cdot \sin t) = -\frac{1}{r \cdot \sin^3 t}.$$

Example 45. Find $\frac{d^3 y}{dx^3}$ if $\begin{cases} x = \ln t, \\ y = t^3 \end{cases}$.

$$y'_x = \frac{dy}{dx} = \frac{(t^3)'}{(\ln t)'} = \frac{3t^2}{\frac{1}{t}} = 3t^3; \quad y''_x = \frac{d^2 y}{dx^2} = \frac{(3t^3)'}{(\ln t)'} = \frac{9t^2}{\frac{1}{t}} = 9t^3;$$

$$y'''_x = \frac{d^3 y}{dx^3} = \frac{(9t^2)'}{(\ln t)'} = \frac{18t}{\frac{1}{t}} = 18t^2.$$

Tasks for independent work

1. Find Feature Differentials Implicitly Defined

a). $x^3 + y^2 + xy = 0$; b). $\arctg \frac{y}{x} - \ln \sqrt{x^2 + y^2} = 0$.

2. Using Leibniz's formula, find a fourth-order derivative for functions

a). $y = x^3 \cdot e^x$; b). $y = x^4 \cdot \ln x$.

3. Find y'' From Function y Implicitly Specified

a). $\arctg y - y + x = 0$; b). $x - y + e^x - e^y = 0$.

4. Find the second-order derivative $\frac{d^2 y}{dx^2}$ of the y function specified in the parametric view

a). $\begin{cases} x = \arctg t, \\ y = \ln(1+t^2) \end{cases}$; b). $\begin{cases} x = \arcsin t, \\ y = \sqrt{1-t^2} \end{cases}$.

List of references recommended for use on this topic.*Main:* 1-2*Additional:* 1-4*Internet resources:* 1-4**Lesson No 10. Basic Theorems of Differential Calculus**

The purpose of the lesson: to master the basic theorems of differential calculus, their geometric meaning, to learn how to apply L'Hôpital's rule when calculating the limits of functions.

Issues for discussion:

1. Fermat's theorem, its geometric meaning.
2. Rolle's theorem, its geometric meaning.
3. Lagrange's theorem, its geometric meaning.
4. Cauchy's theorem on finite increments, its geometric meaning.
5. L'Hôpital's rule for the disclosure of uncertainties.

Tasks solved in the classroom

Example 46. The function $f(x) = -x^2 + 8x$ is differentiable at the segment $[2; 6]$ and at the ends of the segment takes equal values: $f(2) = -4 + 16 = 12$, $f(6) = -36 + 48 = 12$. Therefore, according to Rolle's theorem, there $(2; 6)$ is at least one value of the argument x on the interval at which the derivative $f'(x)$ is converted to zero. Indeed, $f'(x) = -2x + 8$ from where it can be seen that the $x = 4$ derivative $f'(x)$ is reversed to zero. $2 < 4 < 6$

Example 47. Determine the value ξ in the Lagrange formula for the function $f(x) = \ln x$ on the segment $[1; e]$.

According to the condition of the problem, we have: $a=1$, $b=e$, $f(a) = \ln 1 = 0$, $f(b) = \ln e = 1$ $f(x) = \ln x$ $f'(x) = \frac{1}{x}$ $f'(\xi) = \frac{1}{\xi}$. Applying Lagrange's theorem, we will have: $1 - 0 = \frac{1}{\xi} \cdot (e - 1)$, from where $\xi = e - 1$.

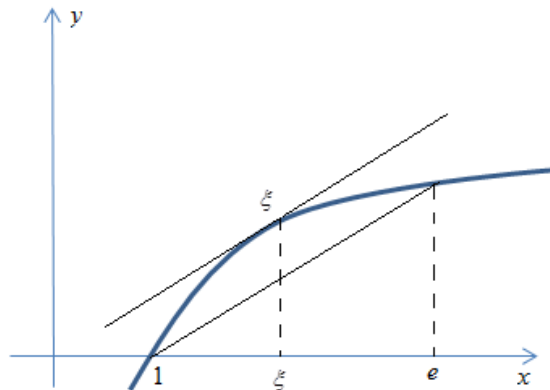


Fig. 25

Example 48. Write the Cauchy formula for the functions $f(x) = x^3$ and $g(x) = x^2$ and define the value ε on the segment $[-2; 1]$.

Under the condition we have $f(a) = (-2)^3 = -8$, $f(b) = 1^3 = 1$, $g(a) = (-2)^2 = 4$, $g(b) = 1^2 = 1$. Further, $f'(x) = 3x^2$, $f'(\varepsilon) = 3\varepsilon^2$, $g'(x) = 2x$, $g'(\varepsilon) = 2\varepsilon$. Substituting in the Cauchy formula, we get $\frac{1 - (-8)}{1 - 4} = \frac{3\varepsilon^2}{2\varepsilon}$, from where $\varepsilon = -2$.

Tasks for independent work

1. To show that between the two roots of the function $f(x) = x^2 - x - 6$ there is a root derivative $f'(x)$.
2. On the parabola $y = x^2 + 3x - 4$, two points are given: $A(-4, 0)$ and $B(2, 6)$. Find point C on the arc AB , at which the tangent is parallel to the chord AB .
3. Determine the value of c in the Lagrange formula for functions
 - a). $f(x) = \ln x$ on the segment $[1, 2]$;
 - b). $f(x) = \arcsin x$ on the segment $[0, 1]$;
 - c). $f(x) = \arctg x$ on the segment $[0, 1]$.
4. Write the Cauchy formula and find c for the specified functions
 - a). $f(x) = \sin x$ and $\varphi(x) = \cos x$ on the segment $\left[0, \frac{\pi}{2}\right]$;
 - b). $f(x) = \cos x$ and $\varphi(x) = x - \sin x$ on the segment $\left[0, \frac{\pi}{2}\right]$.

Tasks solved in the classroom

Example 49. Find the limit $\lim_{x \rightarrow 1} \frac{x^2 - 1 + \ln x}{e^x - e}$.

As can be seen, when an attempt is made to calculate the limit directly, an uncertainty of the form $\left(\frac{0}{0}\right)$. The functions included in the numerator and denominator of the fraction satisfy the

requirements of L'Hôpital's theorem: $f(x) = 2x + \frac{1}{x}$; $g'(x) = ex$; therefore,

$$\lim_{x \rightarrow 1} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow 1} \frac{2x + \frac{1}{x}}{e^x} = \frac{2+1}{e} = \frac{3}{e}.$$

Example 50. Calculate $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{\sin 3x}$.

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{\sin 3x} = \left(\frac{0}{0}\right) = \lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - 1)'}{(\sin 3x)'} = \lim_{x \rightarrow 0} \frac{\frac{1}{2\sqrt{1+x}}}{3 \cos 3x} = \frac{1}{6} \lim_{x \rightarrow 0} \frac{1}{\cos 3x \sqrt{1+x}} = \frac{1}{6}.$$

Example 51. Find the limit $\lim_{x \rightarrow \infty} \frac{\pi - 2 \arctg x}{\frac{3}{e^x - 1}}$.

$$f'(x) = -\frac{2}{1+x^2}; \quad g'(x) = e^{\frac{3}{x}} \cdot \frac{-3}{x^2}; \quad \lim_{x \rightarrow \infty} \left[-\frac{2x^2}{(1+x^2)e^{\frac{3}{x}}(-3)} \right] = \frac{-2}{(0+1) \cdot 1 \cdot (-3)} = \frac{2}{3}.$$

If, when solving the example after applying L'Hôpital's rule, the attempt to calculate the limit again leads to uncertainty, then L'Hôpital's rule can be applied a second time, a third time, and so on, until the result is obtained. Naturally, this is possible only if the newly obtained functions in turn satisfy the requirements of L'Hôpital's theorem.

Example 52. Find the limit $\lim_{x \rightarrow \infty} \frac{x e^{\frac{x}{2}}}{x + e^x}$.

$$f'(x) = e^{\frac{x}{2}} \left(1 + \frac{1}{2}x\right); \quad g'(x) = 1 + e^x;$$

$$f''(x) = \frac{1}{2}e^{\frac{x}{2}} + \frac{1}{2}e^{\frac{x}{2}} + \frac{x}{4}e^{\frac{x}{2}} = \frac{1}{4}e^{\frac{x}{2}}(4+x); \quad g''(x) = e^x;$$

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{4}e^{\frac{x}{2}}(4+x)}{e^x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{4}(4+x)}{\frac{x}{e^2}};$$

$$f'''(x) = \frac{1}{4}; \quad g'''(x) = \frac{1}{2}e^{\frac{x}{2}}; \quad \lim_{x \rightarrow \infty} \frac{1}{\frac{x}{2e^2}} = 0.$$

It should be noted that L'Hôpital's rule is just one of the ways to calculate limits. Often, in a particular example, along with L'Hôpital's rule, some other method can be used (substitution of variables, multiplication, etc.).

Example 53. Find the limit $\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$.

$$f'(x) = e^x + e^{-x} - 2; \quad g'(x) = 1 - \cos x; \quad \lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{1 - \cos x} = \frac{1+1-2}{1-1} = \left(\frac{0}{0}\right) -$$

again there is uncertainty. Let's apply L'Hôpital's rule again: $f''(x) = e^x - e^{-x}$; $g''(x) = \sin x$

$$; \quad \lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{\sin x} = \frac{1-1}{0} = \left(\frac{0}{0}\right) - \text{apply L'Hôpital's rule again } f'''(x) = e^x + e^{-x};$$

$$g'''(x) = \cos x; \quad \lim_{x \rightarrow 0} \frac{e^x + e^{-x}}{\cos x} = \frac{2}{1} = 2.$$

Uncertainties of the form (0^0) ; (1^∞) ; (∞^0) can be revealed using logarithm. Such uncertainties occur when finding the limits of functions of the form $y = [f(x)]^{g(x)}$, $f(x) > 0$ near

the point a at $x \rightarrow a$. To find the limit of such a function, it is enough to find the limit of the function $\ln y = g(x) \ln f(x)$.

Example 54. Find the limit $\lim_{\substack{x \rightarrow 0 \\ x > 0}} x^x$.

Here $y = x^x$, $\ln y = x \ln x$, then

$$\lim_{\substack{x \rightarrow 0 \\ x > 0}} \ln y = \lim_{\substack{x \rightarrow 0 \\ x > 0}} x \ln x = \lim_{\substack{x \rightarrow 0 \\ x > 0}} \frac{\ln x}{\frac{1}{x}} = \lim_{\substack{x \rightarrow 0 \\ x > 0}} \frac{1/x}{-1/x^2} = - \lim_{\substack{x \rightarrow 0 \\ x > 0}} x = 0.$$

$$\text{Therefore } \lim_{\substack{x \rightarrow 0 \\ x > 0}} \ln y = \ln \lim_{\substack{x \rightarrow 0 \\ x > 0}} y = 0; \Rightarrow \lim_{\substack{x \rightarrow 0 \\ x > 0}} y = \lim_{\substack{x \rightarrow 0 \\ x > 0}} x^x = 1.$$

Example 55. Find the limit $\lim_{x \rightarrow \infty} \frac{x^2}{e^{2x}}$.

$f'(x) = 2x$; $g'(x) = 2e^{2x}$; $\lim_{x \rightarrow \infty} \frac{x}{e^{2x}} = \left(\frac{\infty}{\infty} \right)$ - get uncertainty. We apply L'Hôpital's rule again $f''(x) = 2$; $g'(x) = 4e^{2x}$; $\lim_{x \rightarrow \infty} \frac{1}{2e^{2x}} = \frac{1}{\infty} = 0$.

Tasks for independent work

Using L'Hôpital's rule, calculate the limits of functions

- 1). $\lim_{x \rightarrow 1} \frac{x^3 - 2x^2 - x + 2}{x^3 - 7x + 6}$;
- 2). $\lim_{x \rightarrow 1} \frac{\ln x}{x - 1}$;
- 3). $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\ln(1 + 2x)}$;
- 4). $\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$;
- 5). $\lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{1 - \cos 2x}$;
- 6). $\lim_{x \rightarrow 0} \frac{\operatorname{tg} x - \sin x}{x - \sin x}$;
- 7). $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\ln \sin x}{(\pi - 2x)^2}$;
- 8). $\lim_{x \rightarrow 0} \frac{\operatorname{tg} x - \sin x}{\sin^3 x}$;
- 9). $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$;
- 10). $\lim_{x \rightarrow \infty} \frac{x}{\ln(1 + x)}$;
- 11). $\lim_{x \rightarrow 0} \frac{\ln x}{\operatorname{cosec} x}$;
- 12). $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\operatorname{tg} x}{\operatorname{tg} 5x}$;
- 13). $\lim_{x \rightarrow 0} (x \cdot \ln \sin x)$;
- 14). $\lim_{x \rightarrow \frac{\pi}{2}} (\pi - 2x) \cdot \operatorname{tg} x$;
- 15). $\lim_{x \rightarrow \frac{1}{2}} (2x - 1) \cdot \operatorname{tg} \pi x$;
- 16). $\lim_{x \rightarrow 0} \left[\frac{1}{x} - \frac{1}{\ln(1 + x)} \right]$;
- 17). $\lim_{x \rightarrow 0} \left(\operatorname{ctg} x - \frac{1}{x} \right)$;
- 18). $\lim_{x \rightarrow 1} (x - 1)^{\ln x}$;
- 19). $\lim_{x \rightarrow 0} \left(\frac{1}{x} \right)^{\operatorname{tg} x}$;
- 20). $\lim_{x \rightarrow \frac{\pi}{2}} (\sin x)^{\operatorname{tg} x}$.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 11. Using a derivative to study a function

The purpose of the lesson: to learn how to use the apparatus of differential calculus in the study of functions.

Issues for discussion:

1. Signs of increasing and decreasing functions.
2. Function extremum.
3. The smallest and largest values of the function on the segment.
4. Convexity, concavity of the function graph. Inflection points of the function graph.
5. Asymptotes of the function graph.

Tasks solved in the classroom

Example 56. Investigate $y = x^3 + 3x^2 + 3x$ the function of increasing and decreasing.

First we find the derivative of this function $y' = 3x^2 + 6x + 3$, then the intervals of the sign constancy of the function, for which we define the zeros of the function $y' = 3x^2 + 6x + 3 = 3(x+1)^2$, $3(x+1)^2 = 0 \Rightarrow x = -1$ mark them on the set of its definition and place the signs of the derivative in each of the obtained intervals

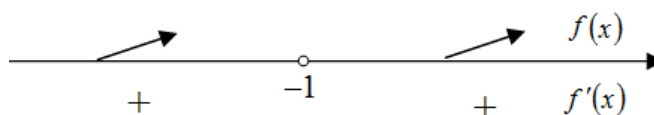


Fig. 26

From the figure it can be seen that the function is increasing over the entire definition domain $y = x^3 + 3x^2 + 3x$.

Example 57. Investigate the extremum of the function $y = \frac{x^3}{3} - x^2 - 3x + 2$.

Let us define the critical points of the function: $y' = x^2 - 2x - 3$, $x^2 - 2x - 3 = 0$, from where $x_1 = -1$ and $x_2 = 3$ - critical points. Let us define the signs of the first derivative near the found critical points. For this purpose, it is convenient to represent the derivative as a product of factors $y' = (x+1)(x-3)$. We see that on the interval $(-\infty; -1)$ the derivative $y' > 0$, on the interval $(-1; 3)$ the derivative $y' < 0$, and on the interval $(3; +\infty)$ the derivative $y' > 0$. Therefore, at the critical point there is $x_1 = -1$ a maximum. At the critical point $x_2 = 3$ The derivative y' changes its sign from minus to plus, at this point the function has a minimum

$$y_{\text{макс}} = y(-1) = -\frac{1}{3} - (-1)^2 + 3 + 2 = 3\frac{2}{3},$$

$$y_{\text{мин}} = y(3) = 9 - 9 - 9 + 2 = -7.$$

Extremes: $A\left(-1; 3\frac{2}{3}\right)$ and $B(3; -7)$.

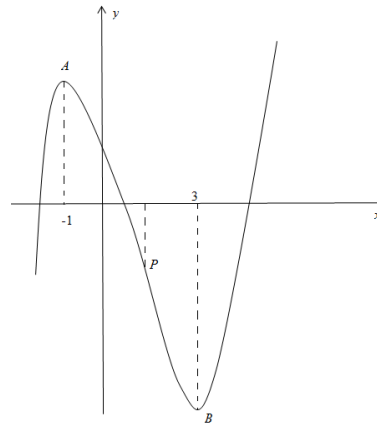


Fig. 27

Example 58. Find the highest and lowest values of the function $y = x^5 - 5x^4 + 5x^3 + 3$ on the segment $[-1, 2]$.

Find the critical points belonging to the interval $(-1, 2)$, and the values of the function at these points $y' = 5x^4 - 20x^3 + 15x^2 = 0$, $x_1 = 0$, $x_2 = 1$, $x_3 = 3$.

The critical point $x_3 = 3$ does not belong to the given segment. Let us calculate the values of the function at two critical points $y(0) = 3$, $y(1) = 4$. Calculate the values of the function at the ends of the given segment $y(-1) = -8$, $y(2) = -5$. Comparing the results obtained, we have: the largest value $M = y(1) = 4$ of the function, and the smallest $m = y(-1) = -8$.

Example 59. Find the inflection points and determine the convexity and concavity intervals of the curve $y = \frac{x^3}{3} - x^2 - 3x + 2$.

Let us find the second derivative of the function $y' = x^2 - 2x - 3$, $y'' = 2x - 2$. Let us define critical points of the second kind $2x - 2 = 0$, from which we can see that the point $x_0 = 1$ divides the entire domain of the definition of this function into two intervals $(-\infty; 1)$ and $(1; +\infty)$; in the first interval the second derivative is negative, and in the second it is positive. Therefore, the point $P\left(1; -\frac{5}{3}\right)$ is the inflection point of the graph of the function. The curve is convex at the interval $(-\infty; 1)$ and concave at the interval $(1; +\infty)$.

Example 60. Find asymptotes and plot the function $y = \frac{x^2 + 2x - 1}{x}$.

- 1) Vertical asymptotes: $y \rightarrow +\infty$, $x \rightarrow 0-0$: $y \rightarrow -\infty$, $x \rightarrow 0+0$, hence $x = 0$ - vertical asymptote.
- 2) Oblique asymptotes:

$$k = \lim_{x \rightarrow \infty} \frac{x^2 + 2x - 1}{x^2} = \lim_{x \rightarrow \infty} \left(1 + \frac{2}{x} - \frac{1}{x^2}\right) = 1,$$

$$b = \lim_{x \rightarrow \infty} (f(x) - x) = \lim_{x \rightarrow \infty} \left(\frac{x^2 + 2x - 1}{x} - x\right) = \lim_{x \rightarrow \infty} \left(\frac{x^2 + 2x - 1 - x^2}{x}\right) =$$

$$= \lim_{x \rightarrow \infty} \left(\frac{2x-1}{x} \right) = \lim_{x \rightarrow \infty} \left(2 - \frac{1}{x} \right) = 2.$$

Thus, the line $y = x + 2$ is an oblique asymptote.
Let's plot the function.

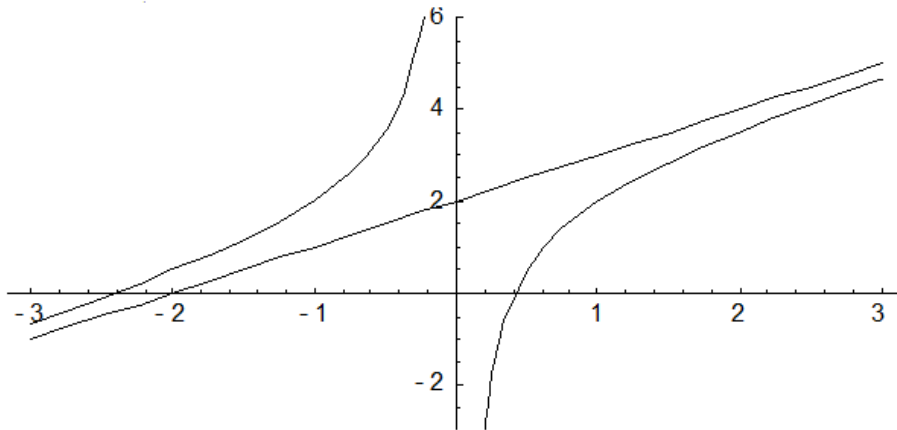


Fig. 28

Example 61. Find asymptotes and plot the function $y = \frac{9x}{9-x^2}$.

The lines $x = 3$ and $x = -3$ are the vertical asymptotes of the curve. We find the oblique

$$\text{asymptotes } k = \lim_{x \rightarrow \infty} \frac{9}{9-x^2} = 0, \quad b = \lim_{x \rightarrow \infty} \frac{9x}{9-x^2} = \lim_{x \rightarrow \infty} \frac{\frac{9}{x}}{\frac{9}{x^2} - 1} = 0.$$

$y = 0$ – horizontal asymptote.

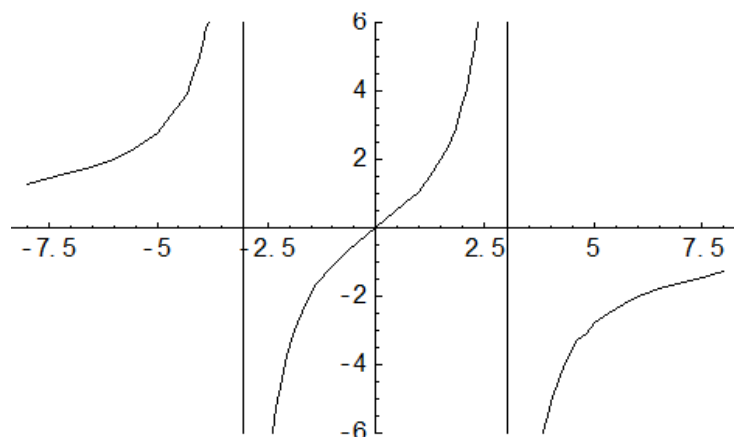


Fig. 29

Example 62. Find asymptotes and plot the function $y = \frac{x^2 - 2x + 3}{x + 2}$.

The line $x = -2$ is the vertical asymptote of the curve.

Let's find the oblique asymptotes:

$$k = \lim_{x \rightarrow \infty} \frac{x^2 - 2x + 3}{x(x+2)} = \lim_{x \rightarrow \infty} \frac{x^2 - 2x + 3}{x^2 + 2x} = \lim_{x \rightarrow \infty} \frac{1 - \frac{2}{x} + \frac{3}{x^2}}{1 + \frac{2}{x}} = 1,$$

$$b = \lim_{x \rightarrow \infty} \left(\frac{x^2 - 2x + 3}{x + 2} - x \right) = \lim_{x \rightarrow \infty} \left(\frac{x^2 - 2x + 3 - x^2 - 2x}{x + 2} \right) =$$

$$= \lim_{x \rightarrow \infty} \frac{-4x + 3}{x + 2} = \lim_{x \rightarrow \infty} \frac{-4 + \frac{3}{x}}{1 + \frac{2}{x}} = -4.$$

So, the line $y = x - 4$ is an oblique asymptote.

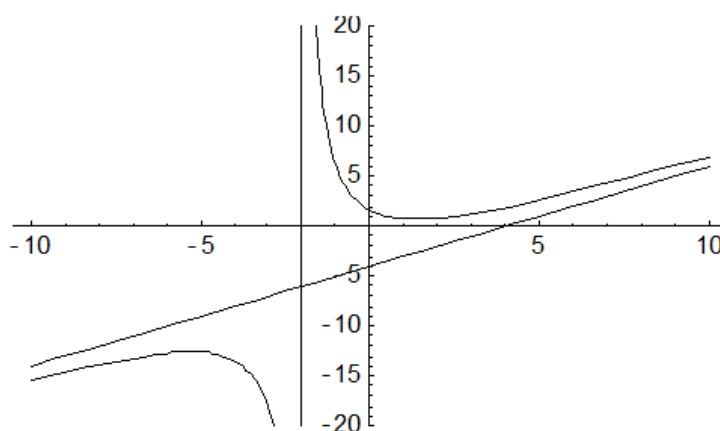


Fig. 30

Example 63. Examine a function $y = \frac{x^3}{x^2 - 1}$ and plot it.

Find the **area of the function definition**: $(-\infty; -1) \cup (-1; 1) \cup (1; \infty)$. In turn, it can be seen that the lines $x = 1$, $x = -1$ are **vertical asymptotes of the curve**, since **the discontinuity points of the second type of function** are the points $x = 1$, $x = -1$. **The range of values of this function** is the interval $(-\infty; \infty)$. If $x = 0$, then $y = 0$, that is, the point $(0; 0)$ belongs to the graph. This function is neither odd nor even.

Let's define **the critical points**, for this we will find the derivative of the function:

$$y' = \frac{3x^2(x^2 - 1) - 2x \cdot x^3}{(x^2 - 1)^2} = \frac{3x^4 - 3x^2 - 2x^4}{(x^2 - 1)^2} = \frac{x^4 - 3x^2}{(x^2 - 1)^2} = \frac{x^2(x^2 - 3)}{(x^2 - 1)^2}.$$

Critical points: $x = 0$; $x = -\sqrt{3}$; $x = \sqrt{3}$; $x = -1$; $x = 1$. Let's establish the **intervals of increasing and decreasing function** by defining the signs of the derived function on the intervals:

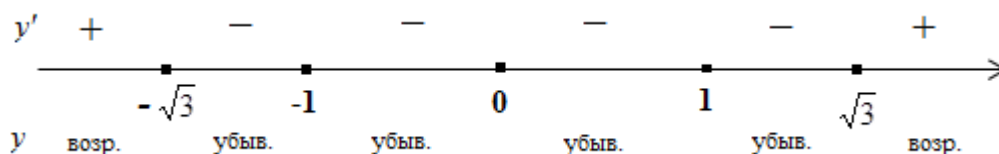


Fig. 31

$-\infty < x < -\sqrt{3}$, $y' > 0$, the function increases
 $-\sqrt{3} < x < -1$, $y' < 0$, function decreases
 $-1 < x < 0$, $y' < 0$, function decreases
 $0 < x < 1$, $y' < 0$, function decreases
 $1 < x < \sqrt{3}$, $y' < 0$, function decreases
 $\sqrt{3} < x < \infty$, $y' > 0$, the function increases

It can be seen that the point $x = -\sqrt{3}$ is **the point of the maximum**, and the point $x = \sqrt{3}$ is **the point of the minimum**. The values of the function at these points are equal to $3\sqrt{3}/2$ and $-3\sqrt{3}/2$, respectively, in other words, **the extreme points** are $A(-\sqrt{3}; 3\sqrt{3}/2)$ and $B(\sqrt{3}; -3\sqrt{3}/2)$.

Next, we look for the second derivative of the function:

$$\begin{aligned}
 y'' &= \frac{(4x^3 - 6x)(x^2 - 1)^2 - (x^4 - 3x^2)4x(x^2 - 1)}{(x^2 - 1)^4} = \\
 &= \frac{(4x^3 - 6x)(x^4 - 2x^2 + 1) - (x^4 - 3x^2)(4x^3 - 4x)}{(x^2 - 1)^4} = \\
 &= \frac{4x^7 - 8x^5 + 4x^3 - 6x^5 + 12x^3 - 6x - 4x^7 + 4x^5 + 12x^5 - 12x^3}{(x^2 - 1)^4} = \\
 &= \frac{2x^5 + 4x^3 - 6x}{(x^2 - 1)^4} = \frac{2x(x^4 + 2x^2 - 3)}{(x^2 - 1)^4} = \frac{2x(x^2 + 3)(x^2 - 1)}{(x^2 - 1)^4} = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}.
 \end{aligned}$$

So, **the critical points of the second kind** are: $x = 0$; $x = -1$; $x = 1$. Determine **the convexity and concavity of the curve** at the intervals:

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y''	-	∞	+	0	-	∞	+
U	iss.	Break	vogn.	kink	iss.	Break	vogn.

$-\infty < x < -1$, $y'' < 0$, convex curve,
 $-1 < x < 0$, $y'' > 0$, concave curve,
 $0 < x < 1$, $y'' < 0$, convex curve,
 $1 < x < \infty$, $y'' > 0$, the curve is concave.

Inflection point $P(0; 0)$.

Vertical **asymptotes** were mentioned above. Now let's find **oblique asymptotes**

$$k = \lim_{x \rightarrow \infty} \frac{x^2}{x^2 - 1} = \lim_{x \rightarrow \infty} \frac{1}{1 - \frac{1}{x^2}} = 1;$$

$$b = \lim_{x \rightarrow \infty} \left(\frac{x^3}{x^2 - 1} - x \right) = \lim_{x \rightarrow \infty} \left(\frac{x^3 - x^3 + x}{x^2 - 1} \right) = \lim_{x \rightarrow \infty} \frac{x}{x^2 - 1} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1 - \frac{1}{x^2}} = 0.$$

Thus, the equation of the oblique asymptote is $y = x$. Let's construct **a** graph of the function based on the results of the study.

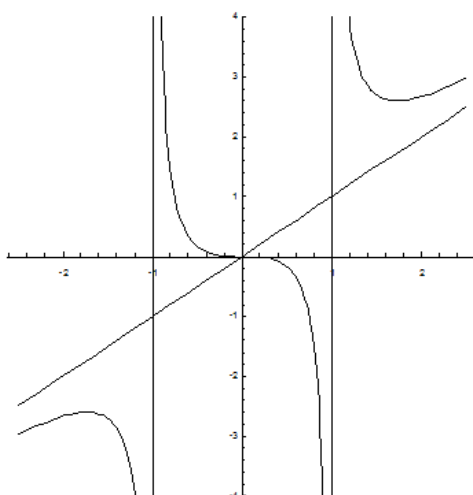


Fig. 32

Tasks for independent work

Determine the intervals of increasing and decreasing of the following functions

a). $y = x^2 - 2x + 3$;

b). $y = x^3 + 3 \cdot x^2 + 3 \cdot x$;

c). $y = -2 \cdot x^3 + 15 \cdot x^2 - 24 \cdot x + 1$;

d). $y = \frac{x}{x-2}$;

e). $y = x + \sin x$;

f). $y = x^2 \cdot e^{-x}$.

7. Find stationary points of functions

a). $y = x^2 - 6x - 4$;

b). $y = 2 \cdot x^3 + 3 \cdot x^2 - 36 \cdot x + 5$.

8. Find Feature Critical Points

a). $y = \sqrt{x}$;

b). $y = (x-1) \cdot \sqrt[3]{x^2}$;

c). $y = \sqrt[3]{2x^2} - \sqrt[3]{(x-2)^2}$;

d). $y = \frac{e^{2x^2}}{x}$.

9. Investigate the extremum of the function

a). $y = 2x^2 - 4x + 5$;

b). $y = \frac{4x}{4+x^2}$;

c). $y = 3 - 2 \cdot \sqrt[3]{x-4}$;

d). $y = \sqrt{e^{x^2} + 8}$;

e). $y = \ln x + \frac{1}{x}$;

f). $y = x - \arctg x$.

10. Find the largest and smallest value of the function on a given segment

a). $y = x^3 e^{x+1}$, $[-4; 0]$;

b). $y = \frac{x}{9-x^2}$, $[-2; 2]$;

c). $y = \frac{2(x^2+3)}{x^2-2x+5}$, $[-3; 3]$;

d). $y = 3 - x - \frac{4}{(x+2)^2}$, $[-1; 2]$;

e). $y = 2x^2 + \frac{108}{x} - 59$, $[2; 4]$;

f). $y = x^2 - 2x + \frac{16}{x-1} - 13$, $[2; 5]$.

11. Find the inflection points and determine the convexity and concavity intervals of the given curves

$$\text{a). } y = x^3 - 6x^2 + 12x + 4; \quad \text{b). } y = \frac{x^3}{x^2 + 12};$$

$$\text{c). } y = \sqrt[3]{x^2 - 1}; \quad \text{d). } y = x^2 \cdot \ln x.$$

12. Find the asymptotes of the specified curves

$$\text{a). } y = x \cdot \operatorname{arctg} x; \quad \text{b). } y = \frac{2x^2 - 9}{x + 2};$$

$$\text{c). } y = \sqrt[3]{x^3 - 6x^2}; \quad \text{d). } y = x \cdot e^{-x}.$$

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 12. Complete Function Study and Graphing

The purpose of the lesson: to conduct a complete study of the function according to the previously worked out scheme and to build its graph.

Issues for discussion:

1. Scheme of a complete study of a function and its construction.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

SECTION III. INTEGRAL CALCULUS FOR FUNCTIONS OF A SINGLE VARIABLE

Lesson No 13. Indefinite integral and its properties

The purpose of the lesson: to master the concepts of the primitive and indefinite integral, the main properties of the indefinite integral.

Issues for discussion:

1. Primitive and indefinite integral.
2. Geometric meaning of an indefinite integral, its properties.
3. Direct integration method (using the properties of an indefinite integral and a table).

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 14. Variable replacement method

The purpose of the lesson: to master the method of integration - the method of replacing a variable (substitution method).

Issues for discussion:

1. Variable replacement method (substitution method).

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 15. Method of integration in parts

The purpose of the lesson: to master the method of integration in parts.

Issues for discussion:

1. Method of integration in parts.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 16. Integration of rational functions

The purpose of the lesson: to master the integration of various rational functions.

Issues for discussion:

1. Integration of elementary fractions.

2. Integration of rationalities by means of decomposition by the sum of simple fractions (method of indeterminate coefficients).

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 17. Integration of irrational functions

The purpose of the lesson: to master the integration of various irrational functions.

Issues for discussion:

1. Integral of the form $\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx$, where n is a natural number.
2. Integration of binomial differentials.
3. Integrals of the form $\int R\left(x, \sqrt{ax^2+bx+c}\right) dx$.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 18. Integration of trigonometric functions

The purpose of the lesson: to master the integration of various trigonometric functions.

Issues for discussion:

1. Integral of the form $\int R(\sin x, \cos x) dx$.
2. Integral of the form $\int R(\sin x, \cos x) dx$, if the function R is odd with respect to $\cos x$.
3. Integral of the form $\int R(\sin x, \cos x) dx$, if the function R is odd with respect to $\sin x$.
4. Integral of the form $\int R(\sin x, \cos x) dx$, function R is even with respect to $\sin x$ and $\cos x$.
5. Integral product of sines and cosines of different arguments.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Semester 2

Lesson No 19. Definite integral and its properties

The purpose of the lesson: to master the concept of a certain integral, to study its basic properties, to learn how to calculate a certain integral using the Newton-Leibniz formula.

Issues for discussion:

1. The concept of a definite integral, its geometric and physical meaning.
2. Basic properties of a certain integral.
3. Relation of a definite integral to an indefinite one, the Newton-Leibniz formula.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 20. Methods for calculating a definite integral

The purpose of the lesson: to master the methods of calculating a certain integral according to the Newton-Leibniz formula, to work out the method of replacing a variable and the method of integrating in parts in a certain integral.

Issues for discussion:

1. Relation of a definite integral to an indefinite one, the Newton-Leibniz formula.
2. A method of replacing a variable in a certain integral.
3. The method of integration in parts in a certain integral.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 21. Applications of a definite integral

The purpose of the lesson: to get acquainted with the applications of a certain integral.

Issues for discussion:

1. Calculate the area of a flat figure.
2. Calculate the arc length of a plane curve.
3. Calculation of the volume of the body of revolution.
4. Calculate the surface area of rotation.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 22. Improper Integrals

The purpose of the lesson: to learn how to calculate improper integrals of the first and second types in case of their convergence or to establish their divergence.

Issues for discussion:

1. Improper integrals of the first type.
2. Improper integrals of the second type.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

SECTION IV. DIFFERENTIAL CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES

Lesson No 23. Functions of several variables. Basic concepts, scope of definition

The purpose of the lesson: to master the definition of the function of several variables, to learn how to perform their general study, to find the field of definition.

Issues for discussion:

1. Define a function of multiple variables.
2. Basic concepts, scope of definition of the function of several variables.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 24. Partial derivatives of functions of several variables

The purpose of the lesson: to learn how to calculate partial derivatives and partial differentials of a function of several variables.

Issues for discussion:

1. Partial derivatives of functions of several variables.
2. Partial differentials of functions of several variables.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 25. Full differential of a function of several variables

The purpose of the lesson: to develop the skills of finding the full differential of the function of several variables.

Issues for discussion:

1. Full differential and its relation to partial derivatives of a function of several variables.
2. Invariance of the form of the full differential. Geometric meaning of the partial derivative and the full differential.

List of references recommended for use on this topic.*Main: 1-2**Additional: 1-4**Internet resources: 1-4***Lesson No 26. Higher Order Partial Derivatives and Differentials**

The purpose of the lesson: to learn how to calculate partial derivatives and complete differentials of higher orders.

Issues for discussion:

1. Partial derivatives and complete differentials of higher orders.

List of references recommended for use on this topic.*Main: 1-2**Additional: 1-4**Internet resources: 1-4***Lesson No 27. Differentiation of complex functions**

The purpose of the lesson: to master the differentiation of complex functions of several variables depending on the type of function (to consider three cases).

Issues for discussion:

1. Differentiation of complex functions of several variables, case 1.
2. Differentiation of complex functions of several variables, case 2.
3. Differentiation of complex functions of several variables, case 3.

List of references recommended for use on this topic.*Main: 1-2**Additional: 1-4**Internet resources: 1-4***Lesson No 28. Extremum of a function of several variables**

The purpose of the lesson: to master the determination of the extremum of the function of several variables, the necessary and sufficient condition for the existence of extremum points.

Issues for discussion:

1. Extremum of a function of several variables.
2. A necessary and sufficient condition for the existence of extremum points of a function of several variables.

List of references recommended for use on this topic.*Main: 1-2**Additional: 1-4**Internet resources: 1-4*

Lesson No 29. The largest and smallest value of the function of several variables

The purpose of the lesson: to master the technique of solving problems to find the largest and smallest value of the function of several variables, to determine the derived function in a given direction, to calculate the gradient of the function of several variables.

Issues for discussion:

1. The highest and lowest values of a function of multiple variables.
2. Directional derivative. Function gradient.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

SECTION V. INTEGRAL CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES

Lesson No 30. Calculation of double integrals by double integration

The purpose of the lesson: to master the definition of a double integral, the necessary condition for its existence, the technique of calculating a double integral in Cartesian coordinates by reducing to a repeated integral.

Issues for discussion:

1. Definition of a double integral, a necessary condition for its existence. Criterion for the existence of a double integral.
2. Classes of integrable functions.
3. Properties of a double integral.
4. Calculation of a double integral in Cartesian coordinates by reducing to a repeated integral.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 31. Calculating the double integral in Cartesian coordinates

The purpose of the lesson: to master the definition of a double integral, the necessary condition for its existence, the technique of calculating a double integral in Cartesian coordinates by reducing to a repeated integral.

Issues for discussion:

1. Properties of a double integral.
2. Calculation of a double integral in Cartesian coordinates by reducing to a repeated integral.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 32. Transition to polar coordinates in double integrals

The purpose of the lesson: to master the replacement of variables under the sign of the double integral, the calculation of the double integral in polar coordinates.

Issues for discussion:

1. Replacement of variables under the sign of a double integral.
2. Calculation of the double integral in polar coordinates.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 33. Geometric Applications of Double Integrals

The purpose of the lesson: to learn how to solve problems for calculating the areas of curvilinear regions, the volumes of bodies of arbitrary shape using a double integral.

Issues for discussion:

1. Geometric applications of the double integral.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 34. Calculation of the triple integral in Cartesian coordinates

The purpose of the lesson: to get acquainted with the concept of "triple integral", to master the technique of calculating it in Cartesian coordinates by reducing it to a threefold integral.

Issues for discussion:

1. Calculation of the triple integral in Cartesian coordinates.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 35. Calculation of the triple integral in cylindrical coordinates

The purpose of the lesson: to work out the calculation of triple integrals in cylindrical coordinates.

Issues for discussion:

1. Triple integral in cylindrical coordinates.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 36. Calculation of the triple integral in spherical coordinates

The purpose of the lesson: to work out the calculation of triple integrals in spherical coordinates.

Issues for discussion:

1. Triple integral in spherical coordinates.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 37. Geometric Applications of Triple Integrals

The purpose of the lesson: to solve geometric problems for the application of triple integrals.

Issues for discussion:

1. Geometric Applications of the Triple Integral.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

SECTION VI. ROWS**Lesson No 38. The concept of a numerical series with positive terms**

The purpose of the lesson: to master the basic concepts and definitions of the theory of numerical series with positive terms.

Issues for discussion:

1. A numerical series and its partial sums, converging and diverging numerical series.
2. The simplest properties of converging series. The remainder of the series.
3. Necessary conditions for the convergence of numerical series.
4. Harmonic series.
5. Cauchy criterion for the convergence of numerical series.

List of references recommended for use on this topic.*Main:* 1-2*Additional:* 1-4*Internet resources:* 1-4**Lesson No 39. Signs of comparison of the convergence of a numerical series, d'Alembert's sign**

The purpose of the lesson: to work out the use of comparison signs, the d'Alembert sign for the study of the convergence of numerical series with positive terms.

Issues for discussion:

1. A sign of comparison of series with non-negative terms.
2. The ultimate feature of comparing rows with non-negative terms.
3. The d'Alembert sign of convergence of series with non-negative terms.

List of references recommended for use on this topic.*Main:* 1-2*Additional:* 1-4*Internet resources:* 1-4**Lesson No 40. Cauchy's radical and integral signs of the convergence of a numerical series**

The purpose of the lesson: to work out the application of radical and integral Cauchy signs of the study of the convergence of numerical series with positive terms.

Issues for discussion:

1. Cauchy's radical sign of convergence of series with non-negative terms.
2. Cauchy's integral criterion of convergence of series with nonnegative terms.

List of references recommended for use on this topic.*Main:* 1-2*Additional:* 1-4*Internet resources:* 1-4**Lesson No 41. Alternating rows. Leibniz's sign**

The purpose of the lesson: to master the concept of an alternating series and to learn how to study its convergence with the help of Leibniz's signs.

Issues for discussion:

1. Alternating rows, Leibniz's sign.
2. Absolute and conditionally converging series.
3. Rearrangement of members in absolutely converging rows.

List of references recommended for use on this topic.*Main:* 1-2*Additional:* 1-4

Internet resources: 1-4

Lesson No 42. Study of the convergence of functional series

The purpose of the lesson: to master the concept of a functional series and to study the study of its convergence.

Issues for discussion:

1. Pointwise and uniform convergence of functional sequences.
2. Cauchy criterion for uniform convergence of a functional sequence.
3. Functional series, the area of their convergence.
4. Uniform convergence of functional series, Weierstrass sign.
5. Limit transition under the sign of the integral. Term integration of functional series.
6. Limit transition under the sign of a derivative. Member-by-term differentiation of functional series.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 43. Study of the convergence of power series

The purpose of the lesson: to master the definition of a power series and to study the issues of studying its convergence.

Issues for discussion:

1. The concept of a power series, Abel's theorem.
2. Interval and radius of convergence of a power series, Cauchy–Hadamard formula.
3. Uniform convergence and continuity of the sum of the power series.
4. Member-by-term integration and differentiation of power series.
5. Problems of the decomposition of a function into a power series, the existence of the decomposition of a function into a Taylor series.
6. Decomposition into power series of elementary functions.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 44. Fourier series decomposition of even and odd functions

The purpose of the lesson: to master the concept of the Fourier trigonometric series and the technique of decomposing even and odd functions into a Fourier series.

Issues for discussion:

1. Fourier trigonometric series. Fourier coefficients.

2. Sufficient signs of Fourier series decomposition.
3. Fourier series decomposition of a nonperiodic function.
4. Fourier series for even and odd functions.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4

Lesson No 45. Fourier series expansion of functions with an arbitrary period

The purpose of the lesson: to solve the problem of Fourier series decomposition of functions with an arbitrary period.

Issues for discussion:

1. Fourier series for functions of any period.

List of references recommended for use on this topic.

Main: 1-2

Additional: 1-4

Internet resources: 1-4